

NONCOMPACT HARMONIC MANIFOLDS

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ABSTRACT. The Lichnerowicz conjecture asserts that all harmonic manifolds are either flat or locally symmetric spaces of rank 1. This conjecture has been proved by Z.I. Szabó [Sz] for harmonic manifolds with compact universal cover. E. Damek and F. Ricci [DR] provided examples showing that in the noncompact case the conjecture is wrong. However, such manifolds do not admit a compact quotient. The classification of all noncompact harmonic spaces is still a very difficult open problem.

In this paper we provide a survey on recent results on noncompact simply connected harmonic manifolds, and we also prove many new results, both for general noncompact harmonic manifolds and for noncompact harmonic manifolds with purely exponential volume growth.

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1. INTRODUCTION

A complete Riemannian manifold X is called harmonic if the harmonic functions satisfy the mean value property, that is, the average on any sphere coincides with its value in the center. Equivalently, for any $p \in X$ the volume density $\theta_p(q) = \sqrt{\det g_{ij}(q)}$ in normal coordinates, centered at any point $p \in X$, is a radial function. In particular, if $c : [0, \infty) \rightarrow X$ is a normal geodesic with $c(0) = p$, the function $f(t) := \theta_p(c(t))$ is independent of c . It is easy to see that all rank 1 symmetric spaces and Euclidean spaces (model spaces) are harmonic. In 1944, A. Lichnerowicz conjectured that conversely every complete harmonic manifold is a model space. He confirmed the conjecture up to dimension 4 [Lic]. It was not before the beginning of the 1990's that general results were obtained. In 1990 Z.I. Szabó [Sz] proved the Lichnerowicz conjecture for compact simply connected spaces. However, not much later, in 1992, E. Damek and F. Ricci [DR] showed that in the noncompact case the conjecture is wrong. They provided examples of homogeneous harmonic spaces which are not symmetric. Nevertheless, in 1995 G. Besson, G. Courtois and S. Gallot [BCG] confirmed the conjecture for manifolds of negative curvature admitting a compact quotient. The proof consisted in a combination of deep rigidity results from hyperbolic dynamics and used besides [BCG] the work of Y. Benoist, P. Foulon and F. Labourie [BFL] and P. Foulon and F. Labourie [FL].

In 2002, A. Ranjan and H. Shah showed [RSh2] that noncompact, simply connected harmonic manifolds of polynomial volume growth are flat. Using a result by Y. Nikolayevski [Ni] showing that the density function f is an exponential polynomial, subexponential volume growth of noncompact simply connected harmonic manifolds implies flatness as well. In 2006, J. Heber [He] proved that among the homogeneous harmonic spaces only the model spaces and the Damek-Ricci spaces

occur. Therefore, it remains to study nonhomogeneous harmonic manifolds of exponential volume growth. In particular, these are spaces without conjugate points and their horospheres have constant mean curvature $h > 0$.

As has been recently observed by the first author [Kn3] the asymptotic nature of the volume growth has a crucial impact on the geometry of the harmonic space. In particular, in [Kn3] it has been proved that purely exponential volume growth, geometric rank 1, Gromov hyperbolicity and the Anosov-property of the geodesic flow (with respect to the Sasaki-metric) are equivalent for noncompact harmonic spaces with $h > 0$. Note that the volume growth is called purely exponential if there exists a constant $c \geq 1$ such that for the volume density f the estimate

$$\frac{1}{c} \leq \frac{f(t)}{e^{ht}} \leq c$$

holds for all $t \geq 1$. In [Kn3] it is also proved that nonpositive curvature or more generally no focal points imply the above conditions. We also note that all examples of noncompact harmonic spaces including the Damek-Ricci spaces have nonpositive curvature. Therefore, the following questions are fundamental in the study of noncompact harmonic spaces:

- (A) Has every non-flat simply connected noncompact harmonic manifold *purely* exponential volume growth?
- (B) Has every non-flat simply connected noncompact harmonic manifold nonpositive curvature?
- (C) Are there *nonhomogeneous* simply connected harmonic manifolds?

In particular, a negative answer to Question (C) implies a positive answer to Question (A) and Question (B). If Question (B) has a positive answer, Question (A) has a positive answer as well but not necessarily vice versa. If Question (A) has an affirmative answer its proof could be considered as a first step into the direction of the classification of all harmonic manifolds. But we like to mention that even under the additional assumption that a non-flat simply connected noncompact manifold (X, g) admits a compact quotient, there is at present no proof that (X, g) has purely exponential volume growth without a further assumption. However, in case a compact quotient exists and the volume growth is purely exponential one can deduce using the above mentioned rigidity results that (X, g) is a symmetric space of negative curvature. In particular, if the answer to question (A) is yes it would solve the classification of harmonic spaces admitting a compact quotient (see [Kn3] for details).

We refer the reader to [Sz], [Krey] and [Be] for well known classical results on harmonic spaces, mostly related to the case of simply connected *compact* harmonic spaces. For informative surveys on *Damek-Ricci spaces*, we refer the reader to [BTV] and [Rou].

The article does not cover recent results on asymptotically harmonic spaces given in [He], [Zi1, Zi2] and [CaSam], nor does it present the integral geometric results for general noncompact spaces given in [PS]. Moreover, the article is complementary to [Kn3]. One of the aims of this article is to present many important other recent results by several authors in a self-contained way. Another aim is the presentation of a number of new results. An overview over these results is given at the beginning of each of the two parts of this article.

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Part 1. General noncompact harmonic manifolds

In this part, we give mostly self-contained presentations of the following topics:

- (1) Nikolayevsky's result [Ni] that the density function of a noncompact harmonic space is an exponential polynomial,
- (2) Ranjan/Shah's result [RSh2] that harmonic spaces with polynomial volume growth are flat, and their integral formula for harmonic functions,
- (3) Zimmer's result [Zi3] that Busemann and Martin boundary coincide on nonflat harmonic spaces, and the calculation of the Radon-Nykodym derivative of the visibility measures.

The material is presented in our own framework and notation, and is often very different from the articles mentioned above. We also present several other new results, amongst them a uniform divergence result for geodesic rays emanating from the same point (Chapter 3), various curvature properties of spheres and horospheres (Chapter 4), and a differential inequality (see (7.5)) for the density function f of every noncompact harmonic space. Moreover, we give an explicit formula for the Green's kernel in terms of the density function, our treatment of visibility measures and their Radon-Nykodym derivative differs considerably from [Zi3], and we discuss representations of bounded harmonic functions in connection with the Martin boundary.

2. THE DENSITY FUNCTION OF HARMONIC MANIFOLDS

The main goal of this chapter is Nikolayevsky's result [Ni] that the density functions of harmonic manifolds are exponential polynomials.

We start with the more general assumption that (X, g) is a complete, simply connected manifold without conjugate points. By a theorem of Cartan-Hadamard, the exponential map $\exp_p : T_p X \rightarrow X$ is a diffeomorphism.

Let us first briefly recall some basic facts on the calculus of Jacobi tensors (see e.g. [Es], [Gre], [Kn2] and [Kn] for more details). Let $c : I \rightarrow X$ be a unit speed geodesic and let $N(c)$ denote the normal bundle of c given by a disjoint union

$$N_t(c) := \{w \in T_{c(t)}X \mid \langle w, \dot{c}(t) \rangle = 0\}.$$

A $(1, 1)$ -tensor along c is a differentiable section

$$Y : I \rightarrow \text{End } N(c) = \bigcup_{t \in I} \text{End}(N_t(c)),$$

i.e., for all orthogonal parallel vector fields x_t along c the covariant derivative of $t \rightarrow Y(t)x_t$ exists. The derivative $Y'(t) \in \text{End}(N_t(c))$ is defined by

$$Y'(t)(x_t) = \frac{D}{dt}(Y(t)x_t).$$

Y is called parallel if $Y'(t) = 0$ for all t . If Y is parallel we have $Y(t)x_t = (Y(0)x)_t$ and, therefore, $\langle Y(t)x_t, y_t \rangle$ is constant for all parallel vector fields x_t, y_t along c . In particular, Y is parallel if and only if Y is a constant matrix with respect to parallel frame field in the normal bundle of c . Therefore, parallel $(1, 1)$ -tensors are also called constant.

The curvature tensor R induces a symmetric $(1, 1)$ -tensor along c given by

$$R(t)w := R(w, \dot{c}(t))\dot{c}(t).$$

A $(1, 1)$ -tensor Y along c is called a Jacobi tensor if it solves the Jacobi equation

$$Y''(t) + R(t)Y(t) = 0.$$

If Y, Z are two Jacobi tensors along c the derivative of the Wronskian

$$W(Y, Z)(t) := Y'^*(t)Z(t) - Y^*(t)Z'(t)$$

is zero and thus, $W(Y, Z)$ defines a parallel $(1, 1)$ -tensor. A Jacobi tensor Y along a geodesic $c : I \rightarrow X$ is called Lagrange tensor if $W(Y, Y) = 0$. The importance of Lagrange tensors comes from the following proposition.

Proposition 2.1. (see, e.g., [Kn3, Prop. 2.1]) *Let $Y : I \rightarrow \text{End } N(c)$ be a Lagrange tensor along a geodesic $c : I \rightarrow X$ which is nonsingular for all $t \in I$. Then for $t_0 \in I$ and any other Jacobi tensor Z along c , there exist constant tensors C_1 and C_2 such that*

$$(2.1) \quad Z(t) = Y(t) \left(\int_{t_0}^t (Y^*Y)^{-1}(s)ds \, C_1 + C_2 \right)$$

for all $t \in I$. Conversely, every tensor of the form (2.1) with Y, C_1, C_2 as above is a Jacobi tensor.

Let SX denote the unit tangent bundle of X with fibres S_pX , $p \in X$ and $\pi : SX \rightarrow X$ be the canonical footpoint projection. For every $v \in SX$, let $c_v : \mathbb{R} \rightarrow X$ denote the unique geodesic satisfying $\dot{c}_v(0) = v$. Define A_v to be the Jacobi tensor along c_v with $A_v(0) = 0$ and $A'_v(0) = \text{id}$. Then the volume of a geodesic sphere $S(p, r)$ of radius r about p is given by

$$\text{vol } S(p, r) = \int_{S_pX} \det A_v(r) d\theta_p(v),$$

where $d\theta_p(v)$ is the volume element of S_pX induced by the Riemannian metric.

Definition 2.2. *Let (X, g) be a complete, simply connected manifold without conjugate points. X is a harmonic manifold if the volume density $\det A_v(t)$ does not depend on $v \in SX$. We call the function*

$$f(t) = \det A_v(t)$$

the density function of the harmonic space X .

Remark. If (X, g) is a harmonic space, we have

$$\text{vol } S(p, r) = \omega_n f(r),$$

where ω_n is the volume of the sphere in the Euclidean space \mathbb{R}^n . Since

$$\frac{f'(r)}{f(r)} = \frac{(\det A_v(r))'}{\det A_v(r)} = \text{tr}(A'_v(r)A_v(r)^{-1})$$

is the mean curvature of the geodesic sphere of radius $r > 0$ about $\pi(v)$ in $c_v(r)$, X is harmonic if and only if the mean curvature of all spheres is a function depending only on the radius.

Proposition 2.3. Let (X, g) be a simply connected manifold without conjugate points and $v \in SX$. Let B_v be the Jacobi tensor along c_v with $B_v(0) = \text{id}$ and $B'_v(0) = 0$. For $t \neq 0$, the tensor

$$Q_v(t) = A_v^{-1}(t)B_v(t)$$

is well defined, since (X, g) has no conjugate points. The tensor Q_v satisfies

$$Q_v(t) - Q_v(s) = - \int_s^t (A_v^* A_v)^{-1}(u) du$$

for $0 < s \leq t$.

Proof. Since A_v is a Lagrange tensor along c_v and nonsingular for $t \neq 0$ ((X, g) has no conjugate points), Proposition 2.1 yields

$$(2.2) \quad B_v(t) = A_v(t) \left(\int_s^t (A_v^* A_v)^{-1}(u) du C_1 + C_2 \right).$$

For $t = s$, we obtain $B_v(s) = A_v(s)C_2$, and therefore $C_2 = Q_v(s)$. Differentiation at $t = s$ yields

$$\begin{aligned} B'_v(s) &= A'_v(s)C_2 + A_v(s)(A_v^* A_v)^{-1}(s)C_1 \\ &= A'_v(s)Q_v(s) + (A_v^*)^{-1}(s)C_1 \\ &= (A'_v(s)A_v^{-1}(s))B_v(s) + (A_v^*)^{-1}(s)C_1 \\ &= (A_v^*)^{-1}(s)(A'_v)^*(s)B_v(s) + (A_v^*)^{-1}(s)C_1. \end{aligned}$$

Here we used that $A'_v(s)A_v^{-1}(s)$ is symmetric since it is the second fundamental form of a geodesic sphere with radius s around p . This implies that

$$A_v^*(s)B'_v(s) = (A'_v)^*(s)B_v(s) + C_1,$$

i.e., the constant tensor C_1 satisfies $C_1 = A_v^*(s)B'_v(s) - (A'_v)^*(s)B_v(s) = -W(A_v, B_v)(s)$. At $s = 0$, we conclude $C_1(0) = -\text{id} \cdot \text{id} = -\text{id}$.

Plugging this into (2.2), we obtain

$$Q_v(t) = A_v^{-1}(t)B_v(t) = - \int_s^t (A_v^* A_v)^{-1}(u) du + Q_v(s).$$

□

Proposition 2.4. *Let $A_{v,s}$ be Jacobitensor along c_v with $A_{v,s}(s) = 0$ and $A'_{v,s}(s) = \text{id}$. Then*

$$(2.3) \quad A_{v,s}(t) = A_v(t) \int_s^t (A_v^* A_v)^{-1}(u) du A_v^*(s).$$

Note that $A_v^*(s)$ in the above formula is a constant tensor.

Proof. $A_{v,s}$ and the right hand side of (2.3) are both Jacobitensors, because of Proposition 2.1. Since

$$A_{v,s}(s) = 0 = A_v(s) \int_s^s (A_v^* A_v)^{-1}(u) du A_v^*(s) = 0$$

and

$$A'_{v,s}(s) = \text{id} = A'_v(s) \cdot 0 + A_v(s)(A_v^* A_v)^{-1}(s)A_v^*(s),$$

both Jacobitensors agree. □

Corollary 2.5. *We have*

$$Q_v(s) - Q_v(t) = A_v^{-1}(t)A_{v,s}(t)(A_v^*)^{-1}(s)$$

Proof. Using Proposition 2.1 and 2.4, we conclude

$$Q_v(s) - Q_v(t) = \int_s^t (A_v^* A_v)^{-1}(u) du = A_v^{-1}(t)A_{v,s}(t)(A_v^*)^{-1}(s).$$

□

Proposition 2.4 leads to the following important result for manifolds without conjugate points:

$\det(Q_v(s) - Q_v(t)) = \frac{\det A_{v,s}(t)}{\det A_v(t) \cdot \det A_v(s)} \quad \text{for } 0 < s \leq t.$

We assume now that (X, g) is a harmonic manifold and are about to present the main result of this chapter. We have $f(t) = \det A_v(t)$ and $\det A_{v,s}(t) = f(t - s)$. Therefore, we obtain

$$\frac{f(t - s)}{f(t)f(s)} = \det(Q_v(s) - Q_v(t)) \quad \text{where } Q_v(s) = A_v^{-1}(s)B_v(s).$$

Our main result (Theorem 2.7 below) follows easily from the following useful fact.

Proposition 2.6. *For a function $\varphi \in C^\infty(\mathbb{R})$, we denote its translation to $s \in \mathbb{R}$ by φ_s and define this function as $\varphi_s(t) = \varphi(t-s)$. Assume that the vector space, spanned by all translates of φ , is finite dimensional. Then φ is an exponential polynomial, i.e., we have*

$$\varphi(t) = \sum_{i=1}^k (p_i(t) \sin(\beta_i t) + q_i(t) \cos(\beta_i t)) e^{\alpha_i t}$$

where p_i, q_i polynomials and $\beta_i, \alpha_i \in \mathbb{R}$.

Proof. Let V be the finite dimensional vector space defined by

$$V := \text{span} \{ \varphi_s \mid s \in \mathbb{R} \}.$$

The derivative of φ can be expressed as the following limit of functions:

$$\begin{aligned} \varphi'(t) &= \lim_{s \rightarrow 0} \frac{\varphi(t) - \varphi(t-s)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\varphi(t) - \varphi_s(t)}{s} \\ &= \left(\lim_{s \rightarrow 0} \frac{\varphi - \varphi_s}{s} \right)(t). \end{aligned}$$

We have $\frac{1}{s}(\varphi - \varphi_s) \in V$ for all $s > 0$ and $\varphi' = \lim_{s \rightarrow 0} \frac{1}{s}(\varphi - \varphi_s)$. Since V is finite dimensional, it is closed and we have $\varphi' \in V$.

Now, the $(\dim V) + 1$ functions $\varphi, \varphi', \dots, \varphi^{(\dim V)} \in V$ must be linear dependent over \mathbb{R} . This shows that φ satisfies a linear ordinary differential equation with constant coefficients and is, therefore, an exponential polynomial. \square

We are now ready to prove our main result.

Theorem 2.7. (see [Ni, Theorem 2]) *Let (X, g) be a harmonic manifold. Then the density function $f(t)$ is an exponential polynomial.*

Proof. As introduced earlier, we define the translations $f_s : \mathbb{R} \rightarrow \mathbb{R}$ by $f_s(t) = f(t-s)$, where f is the density function. Since $Q_v(s) - Q_v(t)$ is a matrix with entries of the form $q(s) - q(t)$, we have

$$\begin{aligned} f_s(t) &= \det(Q_v(s) - Q_v(t)f(t)f(s)) \\ &= \sum_{\alpha=1}^N b_\alpha(s) c_\alpha(t), \end{aligned}$$

for some integer N and with suitable smooth functions b_α, c_α .

In particular, the vector space

$$V := \text{span} \{ f_s \mid s \in \mathbb{R} \} \subset \text{span} \{ c_\alpha \mid 1 \leq \alpha \leq N \}$$

has dimension $\leq N$. Applying Proposition 2.6, we see that $f(t)$ is an exponential polynomial. \square

Remark. Let (X, g) be a harmonic manifold.

(a) The volume of a geodesic ball $B_t(p)$ of radius $t > 0$ in (X, g) can be expressed by

$$\begin{aligned} \text{vol } B_t(p) &= \int_0^t \text{vol } S_u(p) du \\ &= \int_0^t \int_{S_p X} f(u) du = \omega_n \int_0^t f(u) du, \end{aligned}$$

where ω_n is the volume of the $(n-1)$ -dimensional Euclidean unit sphere.

(b) We have $f(t) = (-1)^{\dim X - 1} f(-t)$, since

$$A_v(t) = -A_{-v}(-t) =: C(t)$$

which follows from the fact that both sides are Jacobi-Tensors along c_v with $A_v(0) = C(0) = 0$ and $A'_v(0) = C'(0) = \text{id}$. This implies

$$\begin{aligned} f(t) = \det A_v(t) = \det[-A_{-v}(-t)] &= (-1)^{\dim X - 1} \det A_{-v}(-t) \\ &= (-1)^{\dim X - 1} f(-t). \end{aligned}$$

(c) The quotient $\frac{f'}{f}(r) \geq 0$ is the mean curvature of sphere $S_r(p)$ (with respect to the outward normal vector) in (X, g) , and $\frac{f'}{f}(r)$ is monotone decreasing to the (constant) mean curvature $h \geq 0$ of the horospheres of (X, g) .

Corollary 2.8. Let (X, g) be a harmonic manifold. Then the following properties are equivalent:

- $h = 0$,
- X has polynomial volume growth,
- X has subexponential volume growth.

Proof. Assume $h = 0$. Then we have $\lim_{r \rightarrow \infty} \frac{f'(r)}{f(r)} = 0$. Using l'Hôpital, we obtain

$$\lim_{r \rightarrow \infty} \frac{\log f(r)}{r} = \lim_{r \rightarrow \infty} \frac{f'(r)}{f(r)} = 0.$$

This shows that $f(r)$ has subexponential volume growth.

Assume that $f(r)$ has subexponential volume growth. Since $f(r)$ is an exponential polynomial, this implies

$$|f(r)| \leq C(1+r)^k \quad \forall r \geq 0,$$

with a suitable $C > 0$ and for some $k \geq 0$. Therefore, (X, g) has polynomial volume growth.

Finally, assume that (X, g) has polynomial volume growth. This implies that

$$\lim_{r \rightarrow \infty} \frac{\log f(r)}{r} = 0.$$

On the other hand, we have

$$\lim_{r \rightarrow \infty} \frac{f'(r)}{f(r)} = h.$$

Using l'Hôpital, again, we conclude that

$$0 = \lim_{r \rightarrow \infty} \frac{\log f(r)}{r} = \lim_{r \rightarrow \infty} \frac{f'(r)}{f(r)} = h.$$

This shows that $h = 0$. □

Remark. In chapter 6, we will see that $h = 0$ has an even much stronger implication: A harmonic manifold (X, g) with $h = 0$ is flat.

3. UNIFORM DIVERGENCE OF GEODESICS

In this chapter, we prove that for every distance $d > 0$ and any angle $\alpha > 0$ there exists a $t_0 > 0$, such that any two unit speed geodesics c_1, c_2 starting at the same point and differing by an angle $\geq \alpha$ will diverge uniformly in the sense that $d(c_1(t), c_2(t)) \geq d$ for all $t \geq t_0$. For the proof, we start with the following lemma.

Lemma 3.1. *Let (X, g) be a manifold without conjugate points. Denote by S_v the stable Jacobi tensor along the geodesic c_v defined by $S_v = \lim_{r \rightarrow \infty} S_{v,r}$, where $S_{v,r}$ is the Jacobi tensor along c_v defined by the boundary conditions $S_{v,r}(0) = \text{id}$ and $S_{v,r}(r) = 0$. Then we have*

$$\begin{aligned} \text{(i)} \quad S_v(t) &= A_v(t) \int_t^\infty (A_v^* A_v)^{-1}(u) du, \\ \text{(ii)} \quad S'_v(0) - S'_{v,r}(0) &= \int_r^\infty (A_v^* A_v)^{-1}(u) du. \end{aligned}$$

For a proof see [EOS, pp. 227].

Corollary 3.2. *With the notation in Lemma 3.1, we have*

$$A'_v(t) A_v^{-1}(t) - S'_v(t) S_v^{-1}(t) = A_v^{-1}(t)^* (S'_v(0) - S'_{v,t}(0))^{-1} A_v^{-1}(t).$$

Proof. Evaluating the Wronskian $W(A_v, S_v)(t)$ at $t = 0$ we obtain

$$W(A_v, S_v)(t) = (A'_v)^*(t) S_v(t) - A_v^*(t) S'_v(t) = \text{id}$$

which implies

$$(A_v^*)^{-1}(t) (A'_v)^*(t) - S'_v(t) S_v^{-1}(t) = (A_v^*)^{-1}(t) S_v^{-1}(t).$$

Lemma 3.1 yields

$$S_v^{-1}(t) = (S'_v(0) - S'_{v,t}(0))^{-1} A_v^{-1}(t).$$

Since $B_v = A'_v A_v^{-1}$ is the second fundamental form of spheres centered at $c_v(0)$, B_v is a symmetric operator, i.e.,

$$A'_v(t) A_v^{-1}(t) = (A'_v(t) A_v^{-1}(t))^* = (A_v^*)^{-1}(t) (A'_v)^*(t).$$

Combining these results yields the asserted identity. \square

Proposition 3.3. *Let (X, g) be a noncompact harmonic manifold and f its density function f . Then there exist constants $t_0 > 0$ and $a > 0$ such that*

$$\|A_v(t)\| \geq \frac{a}{\sqrt{\frac{f'(t)}{f(t)} - h}}$$

for all $t \geq t_0$.

Proof. We know that (X, g) has no conjugate points and that the sectional curvature of X is bounded. From the lower bound $-b^2 \leq K_X$ follows (see [Kn, Cor. 2.12 in Section 1.2]) that

$$-b \leq A'_v(t) A_v^{-1}(t) \leq b \coth bt$$

for all $t > 0$, and

$$-b \leq S'_v(t) S_v^{-1}(t) \leq b$$

for all t . Choose $t_0 > 0$ such that $b \coth bt_0 = 2$. Using the identity in Corollary 3.2, we obtain for $t \geq t_0$

$$\begin{aligned} \|(S'_v(0) - S'_{v,t}(0))^{-1}\| &= \|A_v(t)^* (A'_v(t) A_v^{-1}(t) - S'_v(t) S_v^{-1}(t)) A_v(t)\| \\ &\leq \|A_v(t)\|^2 3b. \end{aligned}$$

Since $S'_v(0) - S'_{v,t}(0)$ is positive definite by Lemma 3.1(ii), and

$$\text{tr}(S'_v(0) - S'_{v,t}(0)) = \frac{f'(t)}{f(t)} - h$$

we obtain

$$\begin{aligned} \|(S'_v(0) - S'_{v,t}(0))^{-1}\| &\geq \frac{1}{\|(S'_v(0) - S'_{v,t}(0))\|} \\ &\geq \frac{1}{\text{tr}(S'_v(0) - S'_{v,t}(0))} = \frac{1}{\frac{f'(t)}{f(t)} - h}. \end{aligned}$$

This yields the required estimate. \square

Using this we derive the uniform divergence of geodesics described above.

Corollary 3.4. *Let $c_v : [0, \infty) \rightarrow X$ and $c_w : [0, \infty) \rightarrow X$ be two geodesics with $v, w \in S_p X$. Then*

$$d(c_v(t), c_w(t)) \geq a(t) \angle(v, w)$$

where $a : [0, \infty) \rightarrow [0, \infty)$ is a function (not depending on $p \in X$) with $\lim_{t \rightarrow \infty} a(t) = \infty$.

Proof. Let $c : [0, 1] \rightarrow X$ be a geodesic connecting $c_v(t)$ with $c_w(t)$. Then c is given by

$$c(s) = \exp_p r(s)v(s),$$

where $v(s) \in S_p X$ such that $v(0) = v$, $v(1) = w$ for all $0 \leq s \leq 1$ and $r(0) = r(1) = t$. Then

$$\begin{aligned} \left. \frac{d}{ds} \right|_{s=s_0} c(s) &= D \exp_p(r(s_0)v(s_0))(r'(s_0)v(s_0) + r(s_0)v'(s_0)) \\ &= r'(s_0)c'_{v(s_0)}(r(s_0)) + A_{v(s_0)}(r(s_0))(v'(s_0)). \end{aligned}$$

Since $c'_{v(s_0)}(r(s_0)) \perp A_{v(s_0)}(r(s_0))(v'(s_0))$, we obtain

$$\begin{aligned} \left\| \left. \frac{d}{ds} \right|_{s=s_0} c \right\|^2 &= (r'(s_0))^2 + \|A_{v(s_0)}(r(s_0))v'(s_0)\|^2 \\ &\geq \|A_{v(s_0)}(r(s_0))v'(s_0)\|^2. \end{aligned}$$

If there exists $s_0 \in [0, 1]$ such that $r(s_0) \leq \frac{t}{2}$ then using the triangle inequality we have $d(c_v(t), c_w(t)) \geq t \geq (t/\pi)\angle(v, w)$. If this is not the case, we obtain for all $t > 0$

$$\begin{aligned} d(c_v(t), c_w(t)) = \text{length}(c) &\geq \int_0^1 \|A_{v(s)}(r(s))v'(s)\| ds \\ &\geq \frac{a}{\sqrt{\frac{f'(t/2)}{f(t/2)} - h}} \int_0^1 \|v'(s)\| ds \\ &\geq \frac{a}{\sqrt{\frac{f'(t/2)}{f(t/2)} - h}} \angle(v, w). \end{aligned}$$

The corollary follows now with the choice

$$a(t) = \min \left\{ \frac{t}{\pi}, \frac{a}{\sqrt{\frac{f'(t/2)}{f(t/2)} - h}} \right\}.$$

□

4. CURVATURE PROPERTIES OF SPHERES AND HOROSPHERES

This chapter is devoted to the proof of the following facts about spheres and horospheres:

Proposition 4.1. *Let (X, g) be a noncompact harmonic manifold. Then*

- (A) *There exist constants $C(R_0) > 0$ such that all spheres of radius $r \geq R_0 > 0$ (and horospheres) have sectional curvatures in $[-C(R_0), C(R_0)]$.*

- (B) *All spheres have constant scalar curvature where the constant depends only on the radius.*
- (C) *The horospheres have constant non-positive scalar curvature and the constant does not depend on the horosphere. The constant is zero if and only if X has constant sectional curvature.*

Recall that if A_{v_0} is the Jacobi tensor along the geodesic c_{v_0} with $A_{v_0}(0) = 0$ and $A'_{v_0}(0) = \text{id}$, then $B_{v_0} = A'_{v_0} \cdot A_{v_0}^{-1}$ is the second fundamental form of the sphere centered at $c_{v_0}(0)$. B_{v_0} is symmetric and satisfies the Riccati equation $B'_{v_0} + B_{v_0}^2 + R = 0$. In the proof below we need another notation of the second fundamental form: Let $S_r(p)$ be the sphere of radius $r > 0$ around p , $q \in S_r(p)$, and $v \in T_q X$ be the outward unit normal vector of the sphere $S_r(p)$ at q . Let $v_0 \in T_p X$ be the unit vector such that $v = c'_{v_0}(r)$. We also denote the second fundamental form of $S_r(p)$ at q by $B_{q,v}(r)$, that is, we have $B_{q,v} = B_{v_0}(r)$ (see Figure 1).

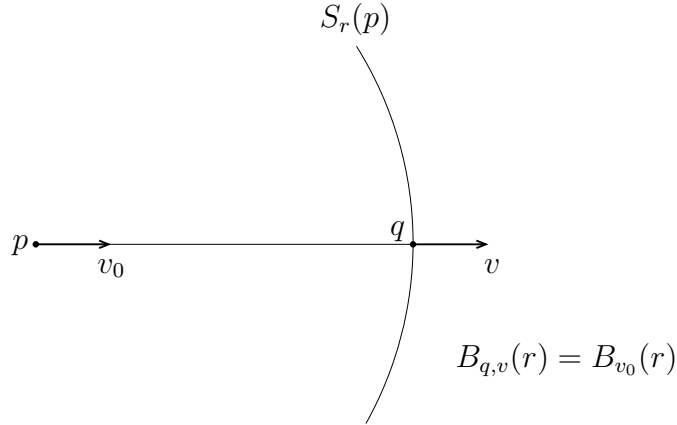


FIGURE 1. Second fundamental forms of spheres

Proof. The Gauss equation for geodesic spheres implies

$$\begin{aligned} \langle R^X(u, w)w, u \rangle &= \langle R^{S_r(p)}(u, w)w, u \rangle \\ &+ \langle u, B_{q,v}(r)w \rangle^2 - \langle B_{q,v}(r)u, u \rangle \cdot \langle B_{q,v}(r)w, w \rangle \quad \forall u, w \in T_q S_r(p). \end{aligned}$$

The sectional curvature of a harmonic space is bounded. Therefore, there exists a constant $C_X > 0$ such that we have for all orthonormal vectors u, w :

$$|K(\text{span}\{u, w\})| = |\langle R(u, w)w, u \rangle| \leq C_X.$$

Since there exists $b > 0$ such that the curvature tensor along c_{v_0} is bounded from below by $-b^2 \leq R(r)$, we have (see [Kn, Cor. 2.12 in Section 1.2])

$$-b \leq B_{q,v}(r) \leq b \coth br.$$

From [RSh3, Prop. 2.1] we also know that $B_{q,v}(r') \leq B_{q,v}(r)$ if $r' \geq r$. Therefore, we can find a constant $C_0(R_0) > 0$ such that $\|B_{q,v}(r)\| \leq C_0(R_0)$ for all $r \geq R_0$. Hence, for $r \geq R_0$, we have

$$\begin{aligned} |K^{S_r(p)}(\text{span}\{u, w\})| &\leq |K^X(\text{span}\{u, w\})| + 2\|B_{q,v}(r)\|^2 \\ &\leq C_X + 2C_0(R_0)^2 = C(R_0). \end{aligned}$$

This shows (A).

For the proof of (B), we consider $u, w \in T_q S_r(p_0)$. Starting with the Gauss equation above for the geodesic sphere $S_r(p)$ and taking trace with respect to u in $T_q S_r(p)$, we obtain

$$\begin{aligned} \text{Ric}^X(w, w) - \langle R^X(v, w)w, v \rangle &= \\ \text{Ric}^{S_r(p)}(w, w) + \|B_{q,v}(r)w\|^2 - \text{tr}[B_{q,v}(r)] \cdot \langle B_{q,v}(r)w, w \rangle &= \\ \text{Ric}^{S_r(p)}(w, w) + \langle B_{q,v}(r)^2 w, w \rangle - \text{tr}[B_{q,v}(r)] \cdot \langle B_{q,v}(r)w, w \rangle. \end{aligned}$$

Taking trace again, now with respect to w in $T_q S_r(p)$, we obtain

$$(n-1)\text{Ric}^X - \text{Ric}^X = \text{scal}^{S_r(p)}(q) + \text{tr}[B_{q,v}(r)^2] - (\text{tr}[B_{q,v}(r)])^2,$$

and therefore

$$(4.1) \quad (n-2)\text{Ric}^X = \text{scal}^{S_r(p)}(q) + \text{tr}[B_{q,v}(r)^2] - (\text{tr}[B_{q,v}(r)])^2$$

The Ricatti equation

$$B'_{v_0}(r) + B_{v_0}^2(r) + R(r) = 0$$

with $R(r)w = R(w, c'_{v_0}(r))c'_{v_0}(r)$ yields, after applying the trace

$$\text{tr}[B'_{v_0}(r)] + \text{tr}[B_{v_0}^2(r)] + \text{Ric}^X = 0.$$

From this and (4.1) we conclude

$$\text{tr}[B'_{v_0}(r)] + (n-1)\text{Ric}^X - \text{scal}^{S_r(p)}(q) + (\text{tr}[B_{v_0}(r)])^2 = 0.$$

Since $\text{tr} B_{v_0}(r) = \frac{f'}{f}(r)$, we obtain

$$\left(\frac{f'}{f}\right)'(r) + (n-1)\text{Ric}^X - \text{scal}^{S_r(p)}(q) + \left(\frac{f'}{f}(r)\right)^2 = 0.$$

Furthermore,

$$\left(\frac{f'}{f}\right)'(r) = \frac{f''f - (f')^2}{f^2}(r)$$

implies

$$(4.2) \quad \frac{f''}{f}(r) + (n-1)\text{Ric}^X - \text{scal}^{S_r(p)}(q) = 0,$$

that is

$$\text{scal}^{S_r(p)}(q) = \frac{f''}{f}(r) + (n-1)\text{Ric}^X.$$

Hence $\text{scal}^{S_r(p)}$ is constant, and the constant depends only on the radius r but not on center p of $S_r(p)$. Therefore, we can write it as scal^{S_r} . This proves (B).

Since the values scal^{S_r} converge to the scalar curvature $\text{scal}^{\mathcal{H}}$ of any horosphere \mathcal{H} as $r \rightarrow \infty$, we know that the scalar curvature of horospheres is constant and the constant does not depend on the horosphere. Moreover, we conclude that $\lim_{r \rightarrow \infty} \frac{f''}{f}(r)$ exists and

$$-\lim_{r \rightarrow \infty} \frac{f''}{f}(r) = (n-1)\text{Ric}^X - \text{scal}^{\mathcal{H}}.$$

Let us denote this limit by α . Our next goal is to determine α . We derive from (4.2) that f satisfies the differential equation

$$f''(r) + ((n-1)\text{Ric}^X - \text{scal}^{S_r})f(r) = 0.$$

Note that we have

$$\lim_{r \rightarrow \infty} (n-1)\text{Ric}^X - \text{scal}^{S_r} = \alpha.$$

Using the result in [Per], we conclude that $\lim_{r \rightarrow \infty} \frac{f'(r)}{f(r)} = \lim_{r \rightarrow \infty} \frac{f'_0(r)}{f_0(r)}$, where f_0 is the solution of

$$f_0''(r) + \alpha f_0(r) = 0.$$

On the other hand, we know that $\lim_{r \rightarrow \infty} \frac{f'(r)}{f(r)} = h$. This shows that $\lim_{r \rightarrow \infty} \frac{f'_0(r)}{f_0(r)} = h$, that is $\alpha = -h^2$. This shows that

$$(4.3) \quad (n-1)\text{Ric}^X - \text{scal}^{\mathcal{H}} = -h^2.$$

The Ricatti equation for orthogonal horospheres along c_{v_0} yields

$$\text{tr}[(B_{v_0}^{\mathcal{H}})'(r)] + \text{tr}[B_{v_0}^{\mathcal{H}}(r)^2] + \text{Ric}^X = 0.$$

Since the trace is linear and $\text{tr}[B_{v_0}^{\mathcal{H}}(r)] = h$, the derivative vanishes and we end up with

$$\text{tr}[B_{v_0}^{\mathcal{H}}(r)^2] + \text{Ric}^X = 0.$$

Using $(n-1) \sum_{j=1}^{n-1} a_j^2 \geq \left(\sum_{j=1}^{n-1} a_j \right)^2$, we conclude

$$(4.4) \quad (n-1) \text{tr}[B_{v_0}^{\mathcal{H}}(r)^2] \geq (\text{tr}[B_{v_0}^{\mathcal{H}}(r)])^2 = h^2,$$

which implies

$$(4.5) \quad (n-1)\text{Ric}^X = -(n-1) \text{tr}[B_{v_0}^{\mathcal{H}}(r)^2] \leq -h^2$$

Combining (4.3) and (4.5), we obtain

$$\text{scal}^{\mathcal{H}} = h^2 + (n-1)\text{Ric}^X \leq h^2 - h^2 = 0.$$

This shows the first part of (C), namely that the scalar curvature of horospheres is a non-positive constant.

For the second part, we can assume that $h > 0$. (In the case $h = 0$, the manifold X is isometric to the flat \mathbb{R}^n , and all its horospheres are

isometric to the flat \mathbb{R}^{n-1} .) The scalar curvature of the horospheres vanishes if and only if $(n-1)\text{Ric}^X = -h^2$, i.e., if the inequality (4.4) holds with equality. The inequality (4.4) holds with equality if and only if $B_{v_0}^{\mathcal{H}}(r)$ is a multiple of the identity for every r . Since $\text{tr}[B_{v_0}^{\mathcal{H}}(r)] = h$, this multiple does not depend on v_0 and r , and we have

$$B_{v_0}^{\mathcal{H}}(r) = \frac{h}{n-1} \text{id}.$$

Looking at the Ricatti equation again

$$(B_{v_0}^{\mathcal{H}})'(r) + (B_{v_0}^{\mathcal{H}}(r))^2 + R(r) = 0,$$

we see that

$$R(r) = -\frac{h^2}{(n-1)^2} \text{id},$$

i.e., the Jacobi operator along c_{v_0} is a fixed constant multiple of the identity, which means that the sectional curvature of X is constant.

Therefore the scalar curvature of horospheres vanishes if and only if X has constant sectional curvature. This yields (C). \square

5. HARMONIC FUNCTIONS WITH POLYNOMIAL GROWTH

For the reader's convenience, let us first recall the notion of sub- and superharmonicity.

Definition 5.1. *Let (X, g) be a Riemannian manifold, $\Delta = \text{div} \circ \text{grad}$ be its Laplacian and $u \in C^2(X)$. u is called subharmonic if $\Delta u \geq 0$. u is called superharmonic if $\Delta u \leq 0$. Obviously, if u is superharmonic then $-u$ is subharmonic, and vice versa. Moreover, a harmonic function is both sub- and superharmonic.*

It is easy to see that subharmonic functions on arbitrary harmonic manifolds satisfy the mean value inequality.

Proposition 5.2. *(Mean Value Inequality) Let (X, g) be a harmonic manifold and $u \in C^2(X)$ be subharmonic. Then we have for all $p \in X$ and $r > 0$,*

$$u(p) \leq \frac{1}{\text{vol } S_r(p)} \int_{S_r(p)} u(q) d\mu_r(q).$$

Proof. Let $\pi_p : C(X) \rightarrow C(X)$ be the radialisation, i.e., $(\pi_p u)(p) = u(p)$ and

$$(\pi_p u)(q) = \frac{1}{\text{vol } S_r(p)} \int_{S_r(p)} u(q) d\mu_r(q)$$

for all $q \in X$ with $d(p, q) = r$. Let u be subharmonic, i.e., $\Delta u \geq 0$. It is well known that π_p and Δ commute. Therefore, we obtain

$$\Delta(\pi_p u) = \pi_p(\Delta u) \geq 0,$$

i.e., $u_0 = \pi_p u$ is again subharmonic. Since u_0 is radial around p , we have $u_0 = F \circ d_p$ and, using (7.1), we have

$$(5.1) \quad 0 \leq \Delta u_0(q) = F'' \circ d_p(q) + \left(\frac{f'}{f} F'\right) \circ d_p(q).$$

Note that F is an even function with $F(0) = u(p)$ and $F'(0) = 0$. Since

$$\frac{1}{\text{vol } S_r(p)} \int_{S_r(p)} u(q) d\mu_r(q) = u_0(q) = F \circ d_p(q),$$

it suffices to show that $F(r) \geq F(0) = u(p)$ for all $r \geq 0$. From (5.1) we conclude

$$(fF')'(r) = f(r)F''(r) + f'(r)F'(r) \geq 0.$$

Integrating over $[0, r]$ and using $f(0) = 0 = F'(0)$, we obtain

$$f(r)F'(r) \geq 0,$$

and since $f(r) > 0$ for $r > 0$, we see that F is monotone increasing on $[0, \infty)$. This shows $F(r) \geq F(0)$, finishing the proof. \square

In the remainder of this chapter, we consider a harmonic manifold (X, g) of polynomial volume growth. The main goal is to prove that the vector space of all harmonic functions of polynomial growth of order $\leq D$ is finite dimensional. A main tool in the proof is the following result for general Riemannian manifolds.

Lemma 5.3. *(compare with [Li, Lemma 28.3]) Let (X, g) be a Riemannian manifold, $p \in X$, and $M \geq 0$ be a non-negative number. Let K be a finite dimensional linear space of functions on X such that for each $u \in K$, there exists a constant $C_u > 0$ such that*

$$(5.2) \quad \int_{B_\rho(p)} |u(x)|^2 dx \leq C_u (1 + \rho)^M \quad \forall \rho \geq 0.$$

Then, for every $\beta > 1, \delta > 0, \rho_0 > 1$, there exists $\rho > \rho_0$ such that the following holds: if $\{u_i\}_{i=1}^k$ with $k = \dim K$ is an orthonormal basis of K with respect to the quadratic form $A_{\beta\rho}(u, v) := \int_{B_{\beta\rho}(p)} u(x)v(x)dx$,

then

$$\sum_{i=1}^k \int_{B_\rho(p)} |u_i(x)|^2 dx \geq k\beta^{-(M+\delta)}.$$

Proof. We assume that the lemma is wrong. Then, for all $\rho > \rho_0$, there exists an orthonormal basis $\{u_i\}$ with respect to $A_{\beta\rho}$ such that

$$\text{tr}_{\beta\rho} A_\rho = \sum_{i=1}^k A_\rho(u_i, u_i) = \sum_{i=1}^k \int_{B_\rho(p)} |u_i(x)|^2 dx < k\beta^{-(M+\delta)}.$$

Note that the trace does not depend on the choice of orthonormal basis $\{u_i\}$. The arithmetic-geometric mean states that $(\det_{\beta\rho} A_\rho)^{\frac{1}{k}} \leq \frac{1}{k} \text{tr}_{\beta\rho} A_\rho$. This implies that

$$(5.3) \quad 0 \leq \det_{\beta\rho} A_\rho \leq \beta^{-k(M+\delta)} \quad \forall \rho \geq \rho_0.$$

Note that for ρ, ρ' we have $\det_{\rho'} A_\rho = \det U_{\rho, \rho'}$, where the endomorphism $U_{\rho, \rho'}$ is defined by $A_\rho(u, v) = A_{\rho'}(U_{\rho, \rho'} u, v)$. This implies that $\det_\rho A_\rho = 1$ and

$$(5.4) \quad \det_{\rho''} A_\rho = (\det_{\rho''} A_{\rho'}) (\det_{\rho'} A_\rho).$$

Applying this identity iteratively to (5.3), we obtain

$$(5.5) \quad \det_{\beta^j \rho} A_\rho \leq \beta^{-jk(M+\delta)} \quad \forall \rho \geq \rho_0,$$

for all positive integers j .

For a fixed orthonormal basis $\{g_i\}$ of K with respect to A_ρ , the growth condition (5.2) implies that there exists a $C > 0$ such that

$$\int_{B_{\rho'(p)}} |g_i(x)|^2 dx \leq C(1 + \rho')^M \quad \forall \rho' \geq 0.$$

This implies that

$$\begin{aligned} 0 \leq \det_\rho A_{\beta^j \rho} &= \det \left(\int_{B_{\beta^j \rho}(p)} g_i(x) g_k(x) dx \right) \\ &\leq C' \sum_{\sigma \in S_k} (1 + \beta^j \rho)^{kM} \leq k! C'' \beta^{jkM} \rho^{kM}, \end{aligned}$$

with suitably chosen constants $C', C'' > 0$. From this we conclude, using again (5.4),

$$(5.6) \quad \det_{\beta^j \rho} A_\rho = (\det_\rho A_{\beta^j \rho})^{-1} \geq \frac{1}{k! C''} \beta^{-jkM} \rho^{-kM}.$$

Combining (5.5) and (5.6), we obtain for all fixed $\rho \geq \rho_0$ and all positive integers j ,

$$\frac{1}{k! C''} \beta^{jk\delta} \leq \rho^{kM}.$$

Since $\beta > 1$, the left hand side tends to infinity as $j \rightarrow \infty$, whereas the right hand side is a constant. This is the desired contradiction. \square

Remark. The case considered in [Li, Lemma 28.3] is a Riemannian manifold (X, g) of polynomial volume growth, i.e.,

$$\text{vol}(B_\rho(p)) \leq C_X(1 + \rho)^\mu \quad \forall \rho \geq 0,$$

and a vector space K such that each $u \in K$ has polynomial growth of degree at most D , i.e., there exists a constant $c_u > 0$ such that

$$|u(x)| \leq c_u(1 + d_p(x))^D \quad \forall x \in X.$$

This implies that

$$\int_{B_\rho(p)} |u(x)|^2 dx \leq C_X c_u (1 + \rho)^{2D+\mu} \quad \forall \rho' \geq 0,$$

and Lemma 5.3 is applicable in this situation with $M = 2D + \mu$.

Now we present the main result.

Theorem 5.4. (see also [LiW, Theorem 4.2] for a more general situation) Let (X, g) be a harmonic manifold of polynomial growth, $p_0 \in X$ and D a positive integer. Let

$$H_D(p_0) := \{\varphi \in C^2(X) \mid \Delta\varphi = 0, \exists c_1, c_2 > 0 : |\varphi(x)| \leq c_1 + c_2 d_{p_0}(p)^D\}$$

Then $\dim H_D(p_0) < \infty$.

Proof. Assume that (X, g) is of polynomial volume growth of degree $\leq \mu$. Henceforth, we fix the constants β, δ, ρ_0 in Lemma 5.3, in particular $\beta = 2$.

Let $\mathcal{H} \subset H_D(p_0)$ be an arbitrary finite dimensional subspace, and denote its dimension by k . We will derive an upper bound on k . Recall that

$$(f, g)_\rho := \int_{B_\rho(p_0)} f(x)g(x) d\text{vol}(x)$$

is a proper inner product for every $\rho > 0$, since a function $f \in \mathcal{H}$ with $(f, f)_\rho = 0$ would have to vanish on $B_\rho(p_0)$. By the unique continuation principle for eigenfunctions (see [Aro]), this would mean $f \equiv 0$.

The remark above shows that inequality (5.2) is satisfied with $M = 2D + \mu$. By Lemma 5.3, we can then find an $r > \rho_0$ such that we have for all orthonormal bases $\{\varphi_i\} \subset \mathcal{H}$ with respect to $(\cdot, \cdot)_{2r}$:

$$(5.7) \quad Ck = C \sum_{j=1}^k \int_{B_{2r}(p_0)} \varphi_j^2(x) dx \leq \sum_{j=1}^k \int_{B_r(p_0)} \varphi_j^2(x) dx,$$

with $C = 2^{-(2D+\mu+\delta)}$.

Since $\sum_{j=1}^k \varphi_j^2$ is subharmonic, i.e., $\Delta(\sum_{j=1}^k \varphi_j^2) \geq 0$, we can apply the maximum principle and obtain

$$\int_{B_r(p_0)} \sum_{j=1}^k \varphi_j^2(x) dx \leq \sum_{j=1}^k \varphi_j^2(q) \cdot \text{vol}(B_r(p_0)),$$

for some $q \in \partial B_r(p_0)$. Since (X, g) is harmonic, we have $\text{vol}(B_r(p_0)) = \text{vol}(B_r(q))$, and we obtain

$$\int_{B_r(p_0)} \sum_{j=1}^k \varphi_j^2(x) dx \leq \sum_{j=1}^k \varphi_j^2(q) \cdot \text{vol}(B_r(q)).$$

Choose an orthogonal transformation $A \in O(k)$ such that the functions $\varphi_l = \sum_{s=1}^k a_{ls} \varphi_s$ satisfy $\psi_2(q) = \cdots = \psi_k(q) = 0$. Then $\sum_{j=1}^k \varphi_j^2(x) = \sum_{j=1}^k \psi_j^2(x)$ for all $x \in X$, and

$$\begin{aligned}
 \int_{B_r(p_0)} \sum_{j=1}^k \varphi_j^2(x) dx &\leq \sum_{j=1}^k \psi_j^2(q) \cdot \text{vol}(B_r(p_0)) = \text{vol}(B_r(p_0)) \cdot \psi_1^2(q) \\
 (5.8) \qquad &= \text{vol}(B_r(q)) \cdot \psi_1^2(q) \leq \int_{B_r(q)} \psi_1^2(x) dx \\
 &\leq \int_{B_{2r}(p_0)} \psi_1^2(x) dx \\
 &= \sum_{j,l=1}^k \int_{B_{2r}(p_0)} a_{1j} a_{1l} \varphi_j(x) \varphi_l(x) dx \\
 (5.9) \qquad &= \sum_{j=1}^k a_{1j}^2 \int_{B_{2r}(p_0)} \varphi_j^2(x) dx = \sum_{j=1}^k a_{1j}^2 = 1.
 \end{aligned}$$

Here, we used the Mean Value Inequality (Proposition 5.2) in (5.8), and orthogonality of the functions φ_j with respect to $(\cdot, \cdot)_{2r}$ in (5.9).

Combining (5.7) with the inequality $\sum_{j=1}^k \int_{B_r(p_0)} \varphi_j^2(x) dx \leq 1$ just derived,

we conclude that $Ck \leq 1$, i.e., $k = \dim \mathcal{H} \leq 1/C = 2^{2D+\mu+\delta}$.

Since $\mathcal{H} \subset H_D(p_0)$ was an arbitrary finite dimensional subspace, $H_D(p_0)$ itself must also be finite dimensional with dimension $\leq 2^{2D+\mu+\delta}$. \square

6. HARMONIC MANIFOLDS WITH $h = 0$ ARE FLAT

Recall from Corollary 2.8 that harmonic manifolds (X, g) with minimal horospheres, i.e., with $h = 0$ (h denotes the mean curvature of the horospheres), must have polynomial volume growth. In this chapter, we present the proof of the much stronger result, due to [RSh2], that $h = 0$ already implies that (X, g) is flat.

The function μ , which we introduce in the following proposition, will play a crucial role in this proof. Let us first collect some general properties of this function.

Proposition 6.1. *Let (X, g) be a general harmonic manifold with density function $f(r)$. Then the function $\mu(r) = \frac{\int_0^r f(s) ds}{f(r)}$ satisfies the following properties:*

- (a) *We have $\mu(0) = 0$, $\mu'(0) = \frac{1}{n}$, $\mu''(0) = 0$, and $\mu'''(0) = \frac{2\text{Ric}^X}{n(n+2)}$.*
- (b) *We have $\mu(r) \geq 0$, for all $r \geq 0$.*

- (c) We have $0 \leq \mu'(r) \leq 1$ for all $r \geq 0$.
 (d) We have $-\mu''(r)\mu(r) \leq \frac{1}{4}$, for all $r \geq 0$.

Proof. **(a)** Recall from [Wi, Chapter 3.6] that

$$f(r) = r^{n-1} \left(1 - \frac{\text{Ric}^X}{6n} r^2 + O(r^4) \right).$$

By integration, we obtain

$$\int_0^r f(t) dt = \frac{r^n}{n} \left(1 - \frac{\text{Ric}^X}{6(n+2)} r^2 + O(r^4) \right).$$

This implies that

$$\begin{aligned} \mu(r) &= \frac{\int_0^r f(s) ds}{f(r)} = \frac{r^n}{nr^{n-1}} \frac{1 - \frac{\text{Ric}^X}{6(n+2)} r^2 + O(r^4)}{1 - \frac{\text{Ric}^X}{6n} r^2 + O(r^4)} \\ &= \frac{r}{n} \left(1 - \frac{\text{Ric}^X}{6(n+2)} r^2 + O(r^4) \right) \left(1 + \frac{\text{Ric}^X}{6n} r^2 + O(r^4) \right) \\ &= \frac{r}{n} \left(1 + \frac{\text{Ric}^X}{6} r^2 \left(\frac{1}{n} - \frac{1}{n+2} \right) + O(r^4) \right) \\ &= \frac{r}{n} + \frac{\text{Ric}^X}{3n(n+2)} r^3 + O(r^5), \end{aligned}$$

which allows us to read off the results of (a).

(b) This follows immediately from $f(r) > 0$ for all $r \geq 0$.

(c) Choose a point $p_0 \in X$. It is straightforward to see that μ satisfies the following differential equation

$$(6.1) \quad \mu'(r) + \frac{f'}{f}(r)\mu(r) = 1,$$

i.e., $\Delta(\mu \circ d_{p_0}) = 1$, which shows that $\mu \circ d_{p_0}$ is subharmonic. Applying the maximum principle to $\mu \circ d_{p_0}$, we see that the restriction of this function to any closed ball around p_0 assumes its maximum at the boundary of this ball. But $\mu \circ d_{p_0}$ is constant along the boundary of any of these balls, and we conclude that

$$\mu'(r) \geq 0 \quad \forall r > 0.$$

On the other hand, using the above differential equation for μ again, as well as $(f'/f)(r) \geq 0$ and $\mu(r) \geq 0$, we obtain

$$\mu'(r) = 1 - \frac{f'}{f}(r)\mu(r) \leq 1.$$

(d) Rewriting (6.1), we obtain

$$\frac{f'}{f}(r) = \frac{1 - \mu'(r)}{\mu(r)},$$

and, consequently, using the fact that $(f'/f)(r)$ converges monotonely decreasing to h ,

$$0 \geq \left(\frac{f'}{f}\right)'(r) = \frac{-\mu''(r)\mu(r) - (1 - \mu'(r))\mu'(r)}{\mu^2(r)}.$$

This implies that

$$-\mu''(r)\mu(r) \leq (1 - \mu'(r))\mu'(r).$$

Since $0 \leq \mu'(r) \leq 1$, we conclude that

$$-\mu''(r)\mu(r) \leq \frac{1}{4}.$$

□

We will also need the following general fact.

Proposition 6.2. *Let (X, g) be a general harmonic manifold and $p_0 \in X$. Let $\varphi \in C^2(X)$ be a function satisfying*

$$\Delta\varphi = c, \quad \varphi(p_0) = 0,$$

for some constant $c \in \mathbb{R}$. Let $g \in C^2(\mathbb{R})$ such that $g \circ d_{p_0} = \pi_{p_0}(\varphi) = \frac{1}{\omega_n} \int_{S_{p_0} X} \varphi(c_w(r)) d\theta_{p_0}(w)$. Then we have

$$g'(r) = c\mu(r).$$

Proof. Note that $g(0) = \varphi(p_0) = 0$ and that g is even, since $\pi_{p_0}(\varphi)$ is radial around p_0 . Therefore, we have $g'(0) = 0$. The function g satisfies the differential equation

$$g''(r) + \frac{f'(r)}{f(r)}g'(r) = c,$$

i.e.,

$$f(r)g''(r) + f'(r)g'(r) = cf(r),$$

which, in turn, transforms into

$$(fg')'(r) = cf(r).$$

Integrating over $[0, r]$, and using $f(0)g'(0) = 0$, leads to

$$g'(r) = c \frac{\int_0^r f(t)dt}{f(r)} = c\mu(r).$$

□

From now on, we assume that (X, g) is a harmonic manifold with $h = 0$, i.e., all its horospheres are minimal. The main step in the proof of flatness is to prove that (X, g) is Ricci flat. We will show that $\text{Ric}^X < 0$ would imply unboundedness of $-\mu''\mu$ from above, in contradiction to Proposition 6.1(d). On the other hand, we cannot have $\text{Ric}^X > 0$, because this would imply compactness of (X, g) , by the

Bonnet-Myers Theorem. But we only consider noncompact harmonic manifolds.

Let $v \in S_{p_0}X$, and b_v be the associated Busemann function with $b_v(p_0) = 0$. Then

$$\Delta b_v = h = 0,$$

i.e., b_v is harmonic. This implies that

$$(6.2) \quad \Delta b_v^2 = 2 \|\text{grad } b_v\|^2 = 2,$$

and we can apply Proposition 6.2 with $\varphi = b_v^2$. Then the function $g \circ d_{p_0} := \pi_{p_0}(b^2)$ satisfies $g'(r) = 2\mu(r)$. Our next goal is to prove that the function g is a very special exponential polynomial.

Proposition 6.3. *g has the following properties:*

- (a) *g is an even function.*
- (b) *We have $0 \leq g(r) \leq r^2$, for all $r \geq 0$.*
- (c) *We have $0 \leq g''(r) \leq 2$, for all $r \geq 0$.*

Proof. (a) Since $\pi_{p_0}(b^2)$ is radial around p_0 , g must be even.

(b) $0 \leq g(r)$ follows from $g(0) = b_v(p_0)^2 = 0$, $g(r) = \frac{1}{2} \int_0^r \mu(t) dt$ and $\mu(r) \geq 0$, for $r \geq 0$. Moreover,

$$g(r) = \frac{1}{\omega_n} \int_{S_{p_0}X} \underbrace{b_v^2(c_w(r))}_{\leq r^2} \leq r^2.$$

(c) This follows from Proposition 6.1(c). □

The next proposition is a key ingredient for the proof that g is an exponential polynomial.

Proposition 6.4. *Let (X, g) be a harmonic manifold with minimal horospheres. Consider the vector space*

$$\mathcal{F} = \{\phi : X \rightarrow \mathbb{R} \mid \exists c, c_1, c_2 > 0 \text{ with } \Delta\phi = c, |\phi(x)| \leq c_1 + c_2 d_{p_0}(x)^2\}.$$

Then \mathcal{F} is finite dimensional.

Proof. We know from Theorem 5.4 that

$$H_2(p_0) = \{\phi : X \rightarrow \mathbb{R} \mid \Delta\phi = 0, \exists c_1, c_2 > 0 \text{ with } |\phi(x)| \leq c_1 + c_2 d_{p_0}(x)^2\}$$

is finite dimensional. The map $\Phi : \mathcal{F} \rightarrow \mathbb{R}, \Phi(\phi) = \Delta\phi$ is linear and $\ker \Phi = H_2(p_0)$. Therefore

$$\dim \mathcal{F} \leq \dim H_2(p_0) + 1 < \infty.$$

□

Next, we introduce the concept of translation of a radial function.

Definition 6.5. *Let $\phi \in C(X)$ be a radial function around p_0 with $\phi(q) = F(d_{p_0}(q))$. The translation of ϕ to another point $p \in X$ is denoted by ϕ_p and defined by*

$$\phi_p(q) = F(d_{p_0}(q)).$$

We now consider the radial function $\phi = \pi_{p_0}(b_v^2)$. Then $\phi = g \circ d_{p_0}$ and $\phi_p = g \circ d_p$. From (6.2) we conclude that

$$\Delta\phi = \Delta\pi_{p_0}(b_v^2) = \pi_{p_0}(\Delta b_v^2) = 2.$$

The translation of ϕ to any point $p \in X$ satisfies also

$$\Delta\phi_p = 2,$$

since Δ , applied to a radial function, is in radial coordinates independent of the centre. Using 6.3(b), we have, for all $p \in X$,

$$|\phi_p(x)| \leq d_p(x)^2 \leq (d_p(p_0) + d_{p_0}(x))^2 \leq 2d_p(p_0)^2 + 2d_{p_0}(x)^2,$$

which shows that $\phi_p \in \mathcal{F}$, for all $p \in X$.

Let $\gamma : \mathbb{R} \rightarrow X$ be a geodesic with $\gamma(0) = p_0$. For a function $\psi \in C^\infty(X)$, we define $\gamma^*\psi \in C^\infty(\mathbb{R})$ via $(\gamma^*\psi)(t) = \psi(\gamma(t))$. Let \mathcal{F} be the finite dimensional vector space introduced in Proposition 6.4. Then the vector space $\tilde{\mathcal{F}} = \gamma^*\mathcal{F} \subset C^\infty(\mathbb{R})$ has also finite dimension.

Note that $g = \gamma^*\phi \in \tilde{\mathcal{F}}$. Let $g_s(t) := g(t - s)$. Then we have, for all $s \in \mathbb{R}$, $g_s = \gamma^*\phi_{\gamma(s)} \in \tilde{\mathcal{F}}$. Applying Proposition 2.6, we see that g is an exponential polynomial. From Proposition 6.3, we know that $0 \leq g(t) \leq t^2$ and that g is even. Therefore, we must have

$$\begin{aligned} g(t) = \sum_{i=1}^{N_1} a_i \cos(\alpha_i t) &+ t \sum_{i=1}^{N_2} b_i \sin(\beta_i t) \\ &+ t^2 \sum_{i=1}^{N_3} c_i \cos(\gamma_i t), \end{aligned}$$

with suitable constants $a_i, b_i, c_i, \alpha_i, \beta_i, \gamma_i \in \mathbb{R}$.

Since $g''(t) \leq 2$ by Proposition 6.3(c), this simplifies to

$$g(t) = \sum_{i=1}^N a_i \cos(\alpha_i t) + ct^2.$$

Differentiating g trice, we obtain

$$\mu''(t) = \frac{1}{2}g'''(t) = \frac{1}{2} \sum_{i=1}^N a_i \alpha_i^3 \sin(\alpha_i t).$$

Now, we assume that $\text{Ric}^X < 0$. Proposition 6.1(a) then tells us that $\mu'''(0) = \frac{2\text{Ric}^X}{n(n+2)} < 0$, which implies that there exist $\delta, r_0 > 0$ such that $\mu''(r_0) = -\delta < 0$. Since μ'' is a finite sum of sines, μ'' is almost periodic, and there exists a sequence $r_k \rightarrow \infty$ such that $\mu''(r_k) = -\delta$ (see, e.g., [Bohr]). This implies that $-\mu(r_k)\mu''(r_k) = \delta\mu(r_k) \rightarrow \infty$, because of

$$\lim_{r \rightarrow \infty} \frac{1}{\mu(r)} = \lim_{r \rightarrow \infty} \frac{f(r)}{\int_0^r f(t)dt} = \lim_{r \rightarrow \infty} \frac{f'(r)}{f(r)} = h = 0.$$

But $\mu(r_k)\mu''(r_k) \rightarrow \infty$ is in contradiction to 6.1(d). This implies that X must be Ricci flat.

Next, we consider the Ricatti equation

$$U'_v(r) + U_v^2(r) + R_v(r) = 0,$$

where $U_v(r)$ (a self-adjoint endomorphism on $c'_v(r)^\perp \subset T_{c_v(r)}X$) denotes the second fundamental form of the horosphere through $c_v(r)$, and centered at $c_v(-\infty)$, and $R_v(r)(w) = R(w, c'_v(r))c'_v(r)$ is the Jacobi operator. Note that $\text{tr } U_v(r) = (\Delta b_v)(c_v(r)) = h = 0$ and, therefore, also $\text{tr } U'_v(r) = (\text{tr } U_v)'(r) = 0$. This implies that, after taking traces,

$$\text{tr } U_v^2(r) = -\text{tr } R_v(r) = -\text{Ric}^X = 0,$$

i.e., $U_v^2(r) = 0$, which implies that $U_v(r) = 0$. Inserting this back into the Riccati equation, we end up with $R_v(r) = 0$, which shows that (X, g) is flat, finishing the proof.

Finally, we like to present another (very strong) criterion, which implies flatness. For non-unit tangent vectors $v \in TX$, let us define the corresponding Busemann functions by $b_v := \|v\|b_{v/\|v\|}$.

Proposition 6.6. *Let (X, g) be a harmonic manifold of dimension n . If the vector space*

$$\text{span}\{b_v \mid v \in T_p X\}$$

is n -dimensional, then the map $F : X \rightarrow \mathbb{R}^n$,

$$F(x) = (b_{e_1}(x), \dots, b_{e_n}(x))$$

is an isometry. In particular, (X, g) is flat.

Proof. Let $\mathcal{B} := \text{span}\{b_v \mid v \in T_p X\}$. Note that the map $F : \mathcal{B} \rightarrow T_p X$, $F(b) = \text{grad } b(p)$ is a bijection, since both vector spaces \mathcal{B} and $T_p M$ have the same dimension and F is surjective, because of $\text{grad } b_v(p) = -v$. Let $v, w \in T_p X$. Since

$$F(b_v + b_w) = -(v + w) = F(b_{v+w}),$$

we conclude that $b_{v+w} = b_v + b_w$.

Let $q \in X$. It is sufficient to show that $DF(q) : T_q X \rightarrow \mathbb{R}^n$ is a linear isometry. Note that

$$DF(q)(w) = (\langle \text{grad } b_{e_1}(q), w \rangle, \dots, \langle \text{grad } b_{e_n}(q), w \rangle).$$

We first show that

$$(6.3) \quad \langle \text{grad } b_{v_1}(q), \text{grad } b_{v_2}(q) \rangle = \langle v_1, v_2 \rangle \quad \forall v_1, v_2 \in T_p X,$$

using $b_{v_1+v_2} = b_{v_1} + b_{v_2}$. Note that $\|b_v(q)\| = \|v\|$ for all $v \in TX$. Using both facts, we obtain

$$\begin{aligned} \|v_1 + v_2\|^2 &= \langle \text{grad } b_{v_1+v_2}(q), \text{grad } b_{v_1+v_2}(q) \rangle \\ &= \langle \text{grad } b_{v_1}(q), \text{grad } b_{v_1}(q) \rangle + \langle \text{grad } b_{v_2}(q), \text{grad } b_{v_2}(q) \rangle \\ &\quad + 2\langle \text{grad } b_{v_1}(q), \text{grad } b_{v_2}(q) \rangle \\ &= \|v_1\|^2 + \|v_2\|^2 + 2\langle \text{grad } b_{v_1}(q), \text{grad } b_{v_2}(q) \rangle. \end{aligned}$$

This shows (6.3). Therefore $\{\text{grad } b_{e_i}(q) \mid 1 \leq i \leq n\}$ are an orthonormal basis of $T_q X$. This implies that

$$\|DF(q)(w)\|^2 = \sum_{i=1}^n \langle \text{grad } b_{e_i}(q), w \rangle^2 = \|w\|^2.$$

□

Remark. *It is tempting to use Proposition 6.6, to cook up an alternative proof of the fact that $h = 0$ implies flatness of (X, g) . But our attempt to do this, falls short. Nevertheless, let us see, how far we get: In the case $h = 0$, the harmonic manifold (X, g) has polynomial growth, by Corollary 2.8. Moreover, we have $\Delta b_v = 0$, and therefore*

$$\text{span}\{b_v \mid v \in TX\} \subset H_1(p_0).$$

Theorem 5.4 tells us that this span is finite dimensional. In view of Proposition 6.6, we would like to show that

$$\dim(\text{span}\{b_v \mid v \in T_p X\}) = n.$$

But we do not know how to derive such a precise result on the dimension of this space.

7. SPECIAL EIGENFUNCTIONS OF GEODESIC SPHERES

Let (X, g) be an arbitrary harmonic manifold with reference point p_0 . Ranjan/Shah introduce in [RSh2] an interesting family of functions φ_v on $X \setminus \{p_0\}$. It turns out that these functions, restricted to the geodesic spheres $S_r(p_0)$, are eigenfunctions of the Laplacian $\Delta^{S_r(p_0)}$ for all radii $r > 0$. These eigenfunctions have just two nodal domains, both with half the volume of $S_r(p_0)$, and it is natural to assume that φ_v are eigenfunctions to the smallest non-trivial eigenvalue of $\Delta^{S_r(p_0)}$ (see [RSh2, p. 690, Remark (iii)]). But there is currently no proof of this assumption.

We first define $w_{p_0}(q) \in S_{p_0} X$ by

$$\exp_{p_0}(d_{p_0}(q)w_{p_0}(q)) = q,$$

where $d_{p_0}(q) = d(p_0, q)$.

Proposition 7.1. *(see [RSh2, Section 4]) For each $v \in T_{p_0} X$, let $\varphi_v : X \setminus \{p_0\} \rightarrow \mathbb{R}$ be the function*

$$\varphi_v(q) = \langle v, w_p(q) \rangle.$$

Then the restriction of φ to the geodesic sphere $S_r(p_0) \subset X$ is an eigenfunction of the Laplacian $\Delta^{S_r(p_0)}$. The corresponding eigenvalue is $-\left(\frac{f'}{f}\right)'(r) > 0$.

Proof. Let $\gamma : (-\epsilon, \epsilon) \rightarrow X$ be a smooth curve with $\gamma'(0) = v \in T_{p_0}X$. Then we have

$$\begin{aligned} \left. \frac{d}{ds} \right|_{s=0} d_q(\gamma(s)) &= \langle \text{grad } d_q(p), \gamma'(0) \rangle \\ &= -\langle w_{p_0}(q), v \rangle = -\varphi_v(q). \end{aligned}$$

Using

$$(7.1) \quad \Delta_q^X(F \circ d_p)(q) = F'' \circ d_p(q) + \left(\frac{f'}{f} F' \right) \circ d_p(q)$$

and the chain rule, we conclude

$$\begin{aligned} \Delta^X \varphi_v(q) &= -\Delta_q \left(\left. \frac{d}{ds} \right|_{s=0} d_q(\gamma(s)) \right) = -\left. \frac{d}{ds} \right|_{s=0} (\Delta_q d_{\gamma(s)})(q) \\ &= -\left. \frac{d}{ds} \right|_{s=0} \frac{f'}{f} (d(\gamma(s), q)) = -\left(\frac{f'}{f} \right)' (d_{p_0}(q)) (-\varphi_v(q)). \end{aligned}$$

This implies

$$\Delta^X \varphi_v(q) = \left(\frac{f'}{f} \right)' (d_{p_0}(q)) \varphi_v(q).$$

Since the radial derivatives of $\varphi_v(q)$ with respect to the centre p_0 vanish, we have for all $q \in S_r(p_0)$

$$\Delta^X \varphi_v(q) = \Delta^{S_r(p_0)} \varphi_v(q),$$

i.e.,

$$\Delta^{S_r(p_0)} \varphi_v(q) - \left(\frac{f'}{f} \right)' (r) \varphi_v(q) = 0.$$

□

Remark. Let (X, g) be a harmonic manifold of dimension n with reference point p_0 .

(a) The eigenspace \mathcal{E} of $\Delta^{S_r(p_0)}$ to the eigenvalue $-\left(\frac{f'}{f} \right)' (r)$ has dimension $\geq n$. This follows from the fact that the map $T_{p_0}X \rightarrow \mathcal{E}$, $v \mapsto \varphi_v$ is linear and injective.

(b) We have the asymptotics $-\left(\frac{f'}{f} \right)' (r) \rightarrow 0$ as $r \rightarrow \infty$, see Chapter 4. It would be interesting to find out whether $-\left(\frac{f'}{f} \right)' (r) > 0$ is the smallest non-zero eigenvalue of $S_r(p_0)$.

(c) After the canonical identification of $S_r(p_0)$ with $S_{p_0}X$ via the exponential map and pullback of the Riemannian metric on $S_r(p_0) \subset X$, we obtain a family of Riemannian manifolds $(S_{p_0}X, g_r)$ such that the Laplace eigenfunction φ_v is a spherical harmonic of degree 1. In general, it is unlikely that spherical harmonics of higher degree are also Laplace eigenfunctions, but these spherical harmonics can be used to obtain orthonormal bases of the Hilbert spaces $L^2(S_{p_0}X, g_r)$, $r > 0$, since the Riemannian measures of $(S_{p_0}X, g_r)$ are multiples of the Euclidean measure of $S_{p_0}X \subset T_{p_0}X$ (viewed as a round unit sphere).

Let (X, g) be a harmonic manifold and $v \in T_p X$. Consider the function $\varphi_v : X \rightarrow \mathbb{R}$ defined by

$$\varphi_v(q) = \langle v, w_p q \rangle$$

where $q = \exp_p(d(p, q)w_p(q))$. Since in the radial direction the function is constant we have for $q = c_w(r)$: $\text{grad } \varphi_v(q) \in T_q S_{d(p, q)}(p)$. We want to calculate the gradient of φ_v :

Take a curve $\alpha : (-\epsilon, \epsilon) \rightarrow S_p X$ with $\alpha(0) = w$. Then

$$\varphi_v(c_{\alpha(s)}(r)) = \langle v, \alpha(s) \rangle$$

This implies

$$\left\langle \text{grad } \varphi_v(q), \frac{\partial}{\partial s} \Big|_{s=0} c_{\alpha(s)}(r) \right\rangle_q = \left\langle v, \frac{\partial}{\partial s} \Big|_{s=0} \alpha(s) \right\rangle_p$$

Let A_w be orthonormal Jacobitensor along c_w with $A_w(0) = 0$ and $A'_w(0) = \text{id}$. Then $\det(A_w(r)) = f(r)$ and

$$\left\langle \text{grad } \varphi_v(q), A_w(r) \frac{\partial}{\partial s} \Big|_{s=0} \alpha(s) \right\rangle_q = \left\langle v, \frac{\partial}{\partial s} \Big|_{s=0} \alpha(s) \right\rangle_p$$

and therefore

$$\begin{aligned} \left\langle A_w^*(r) \text{grad } \varphi_v(q), \frac{\partial}{\partial s} \Big|_{s=0} \alpha(s) \right\rangle_p &= \left\langle v, \frac{\partial}{\partial s} \Big|_{s=0} \alpha(s) \right\rangle_p \\ &= \left\langle v - \langle v, w \rangle w, \frac{\partial}{\partial s} \Big|_{s=0} \alpha(s) \right\rangle_p. \end{aligned}$$

Since this holds for all $\frac{\partial}{\partial s} \Big|_{s=0} \alpha(s)$ we have

$$(7.2) \quad A_w^*(r) \text{grad } \varphi_v(q) = v - \langle v, w \rangle w$$

and

$$\text{grad } \varphi_v(q) = (A_w^*(r))^{-1}(v - \langle v, w \rangle w).$$

This yields

$$\|\text{grad } \varphi_v(q)\|^2 = \langle A_w^{-1}(r)(A_w^*(r))^{-1}(v - \langle v, w \rangle w), v - \langle v, w \rangle w \rangle.$$

Since φ_v is eigenfunction with $\Delta^{S_r(p)} \varphi_v = (\frac{f'}{f}(r)) \varphi_v$, we have

$$\frac{\int_{S_r(p)} \|\text{grad } \varphi_v\|^2 d\mu_r}{\int_{S_r(p)} \varphi_v^2 d\mu_r} = - \frac{\int_{S_r(p)} \Delta^{S_r(p)} \varphi_v(q) \varphi_v(q) d\mu_r(q)}{\int_{S_r(p)} \varphi_v^2(q) d\mu_r(q)} = - \left(\frac{f'}{f} \right)'(r).$$

On the other hand

$$\int_{S_r(p)} \varphi_v^2 d\mu_r = \int_{S_p X} \langle v, w \rangle^2 f(r) d\theta_p(w) = f(r) \frac{w_{n-1}}{n} \langle v, v \rangle$$

and

$$\int_{S_r(p)} \|\text{grad } \varphi_v\|^2 d\mu_r = f(r) \int_{S_p X} \langle A_w^* A_w \rangle^{-1}(r) \langle v - \langle v, w \rangle w, v - \langle v, w \rangle w \rangle d\theta_p(r)$$

implies

$$-\left(\frac{f'}{f}\right)'(r) \langle v, v \rangle = \frac{n}{w_{n-1}} \int_{S_p X} \langle (A_w^* A_w)^{-1}(r) \langle v - \langle v, w \rangle w, v - \langle v, w \rangle w \rangle d\theta_p$$

for all $v \in T_p X$. Define the symmetric endomorphism

$$H_w(r)u = \begin{cases} (A_w^*(r)A_w(r))^{-1}(u) & \text{if } u \perp w, \\ 0 & \text{if } u \in \text{span}\{w\}. \end{cases}$$

We obtain the identity

$$\frac{n}{\omega_n} \int_{S_p X} \langle H_w(r)v, v \rangle d\theta_p(w) = -\left(\frac{f'}{f}\right)'(r) \langle v, v \rangle.$$

Since both sides are symmetric linear forms, we conclude that

$$\frac{n}{\omega_n} \int_{S_p X} \langle H_w(r)u, v \rangle d\theta_p(w) = -\left(\frac{f'}{f}\right)'(r) \langle u, v \rangle$$

and

$$(7.3) \quad \frac{n}{\omega_n} \int_{S_p X} H_w(r)(\cdot) d\theta_p(w) = -\left(\frac{f'}{f}\right)'(r) \text{id}_{T_p X}.$$

Using the arithmetic-geometric mean, we have

$$(7.4) \quad \left(\frac{1}{f^2(r)}\right)^{\frac{1}{n-1}} = \det(H_w(r)|_{w^\perp})^{\frac{1}{n-1}} \leq \frac{1}{n-1} \text{tr} H_w(r).$$

Taking traces in (7.3)

$$\frac{n}{\omega_n} \int_{S_p X} \text{tr} H_w(r)(\cdot) d\theta_p(w) = -n \left(\frac{f'}{f}\right)'(r),$$

and using (7.4) yields the inequality:

$$\frac{n(n-1)}{\omega_n} \int_{S_p X} \left(\frac{1}{f^2(r)}\right)^{\frac{1}{n-1}} d\theta_p(w) \leq -n \left(\frac{f'}{f}\right)'(r).$$

Therefore

$$(n-1) \frac{1}{f^{\frac{2}{n-1}}(r)} \leq -\left(\frac{f'}{f}\right)'(r).$$

or equivalently

$$(7.5) \quad (n-1) \leq -f^{\frac{2}{n-1}}(r) \left(\frac{f'}{f} \right)'(r).$$

The density function $f(r)$ of every noncompact harmonic space satisfies the differential inequality (7.5). In cases $f(r) = r^{n-1}$ or $f(r) = \sin h^{n-1}(r)$ we have equality in (7.5).

Proposition 7.2. *The eigenvalue $-\left(\frac{f'}{f}\right)'(r)$ of the Laplacian on geodesic spheres of radius r tends to zero as r tends to ∞ . If $-\left(\frac{f'}{f}\right)'(r)$ tends to zero on an exponential rate, the volume growth of (X, g) is purely exponential.*

Proof. The first assertion follows from (4.3) since

$$\lim_{r \rightarrow \infty} \left(-\frac{f'}{f} \right)'(r) = \lim_{r \rightarrow \infty} \frac{f''}{f}(r) - \frac{(f')^2}{f^2}(r) = 0.$$

To prove the second claim consider the function $a : [0, \infty) \rightarrow [0, \infty)$ defined by $f(r) = e^{hr}a(r)$. Since

$$\frac{f'}{f}(r) = \frac{e^{hr}(ha(r) + a'(r))}{e^{hr}a(r)} = h + \frac{a'}{a}(r),$$

we have

$$\left(-\frac{f'}{f} \right)'(r) = \left(-\frac{a'}{a} \right)'(r).$$

Since $\frac{f'}{f}(r)$ is monotonically decreasing and converging to h , $\frac{a'}{a}(r)$ is monotonically decreasing and converging to 0. If $-\left(\frac{f'}{f}\right)'(r)$ tends to zero at an exponential rate we have constants $c > 0$ and $r_0 > 0$ such that

$$\left(-\frac{f'}{f} \right)'(r) \leq e^{-cr}$$

for all $r \geq r_0$. Hence, for $r \geq r_0$ we obtain

$$\frac{a'}{a}(r) = \int_r^\infty e^{-cs} ds = \frac{1}{c} e^{-cr},$$

and therefore

$$\frac{a(r)}{a(r_0)} \leq \frac{1}{c^2} (e^{-cr_0} - e^{-cr}) < \frac{1}{c^2} e^{-cr_0},$$

which yields the second assertion. \square

8. AN INTEGRAL FORMULA FOR SUBHARMONIC FUNCTIONS

Let (X, g) be a harmonic manifold. Our main goal in this chapter is an explicit integral formula for the derivative of harmonic functions which was first presented by Ranjan and Shah in [RSh2, Theorem 2.1]. Their formula generalises to sub- and superharmonic functions. We present a derivation in this more general setting. Our derivation differs from the proof in [RSh2].

Assume $u \in C^2(X)$ is subharmonic (i.e., $\Delta u \geq 0$). We present the derivation in this case. The subharmonic case is derived by replacing u by $-u$. Let $\psi = u \cdot \varphi_v$ and $q = c_w(r)$ for a suitable choice of $w \in S_p X$ and $r > 0$. Then

$$\begin{aligned} \Delta \psi(q) &= \frac{d^2}{dr^2} u(c_w(r)) \langle v, w \rangle + \frac{f'}{f}(r) \frac{d}{dr} u(c_w(r)) \langle v, w \rangle + \Delta^{S_r(p)}(u \cdot \varphi_v)(q) \\ &\geq -(\Delta^{S_r(p)} u)(q) \varphi_v(q) + \Delta^{S_r(p)}(u \cdot \varphi_v)(q). \end{aligned}$$

Integrations over $S_r(p)$ yields

$$\begin{aligned} \int_{S_r(p)} \Delta \psi(q) d\mu_r(q) &= f(r) \frac{d^2}{dr^2} \int_{S_p X} u(c_w(r)) \langle v, w \rangle d\theta_p(w) + \\ &\quad + f'(r) \frac{d}{dr} \int_{S_p X} u(c_w(r)) \langle v, w \rangle d\theta_p(w) \\ &\geq \int_{S_r(p)} (\Delta^{S_r(p)} u)(q) \cdot \varphi_v(q) d\mu_r(q) \\ &= - \int_{S_r(p)} u(q) \left(\frac{f'}{f} \right)'(r) \varphi_v(q) d\mu_r(q) \\ &= -f(r) \left(\frac{f'}{f} \right)'(r) \int_{S_p X} u(c_w(r)) \langle v, w \rangle d\theta_p(w). \end{aligned}$$

Introducing $g(r) = \int_{S_p X} u(c_w(r)) \langle v, w \rangle d\theta_p(w)$, we obtain

$$f(r)g''(r) + f'(r)g'(r) + f(r) \left(\frac{f'}{f} \right)'(r)g(r) \geq 0.$$

Since $f > 0$, we can divide by f

$$g''(r) + \frac{f'}{f}(r)g'(r) + f(r) \left(\frac{f'}{f} \right)'(r)g(r) \geq 0,$$

and simplify the result to

$$(8.1) \quad g''(r) + \left(\frac{f'}{f} g \right)'(r) \geq 0.$$

We have the initial conditions

$$g(0) = u(p) \int_{S_p X} \langle v, w \rangle d\theta_p(w) = 0$$

and

$$g'(0) = \int_{S_p X} \langle \text{grad } u(p), w \rangle \langle v, w \rangle d\theta_p(w) = \frac{\omega_n}{n} \langle \text{grad } u(p), v \rangle.$$

Next, we integrate (8.1) over $[0, r]$ and obtain

$$g'(r) - g'(0) + \left(\frac{f'}{f} g \right) (r) - \left(\frac{f'}{f} g \right) (0) \geq 0.$$

Since

$$\begin{aligned} \lim_{t \rightarrow 0} \left(\frac{f'}{f} g \right) (t) &= \lim_{t \rightarrow 0} \frac{n-1}{t} g(t) = (n-1) \lim_{t \rightarrow 0} \frac{g(t) - g(0)}{t} \\ &= (n-1) g'(0) = \frac{n-1}{n} \omega_n \langle \text{grad } u(p), v \rangle, \end{aligned}$$

we conclude

$$g'(r) + \left(\frac{f'}{f} g \right) (r) \geq n g'(0) = \omega_n \langle \text{grad } u(p), v \rangle.$$

Multiplying by $f > 0$, we obtain

$$(fg)'(r) = (fg')(r) + (f'g)(r) \geq \omega_n \langle \text{grad } u(p), v \rangle f(r).$$

Integrating, again, over $[0, r]$ and using $f(0) = 0 = g(0)$ leads to

$$(fg)(r) = (fg)(r) - (fg)(0) \geq \omega_n \langle \text{grad } u(p), v \rangle \int_0^r f(t) dt,$$

i.e.,

$$\begin{aligned} \langle \text{grad } u(p), v \rangle &\leq \frac{f(r)}{\omega_n \int_0^r f(t) dt} \int_{S_p X} u(c_w(r)) \langle v, w \rangle d\theta_p(w) \\ &= \frac{1}{\text{vol}(B_r(p))} \int_{S_r(p)} u(q) \varphi_v(q) d\mu_r(q). \end{aligned}$$

This proves the following result:

Theorem 8.1. *Let (X, g) a harmonic manifold. If u is subharmonic on X (i.e., $\Delta u \geq 0$) and $v \in T_p X$, then*

$$\langle \text{grad } u(p), v \rangle \leq \frac{1}{\text{vol}(B_r(p))} \int_{S_r(p)} u(q) \varphi_v(q) d\mu_r(q)$$

for all $r > 0$.

Corollary 8.2. *Let (X, g) be a harmonic manifold with subexponential volume growth. If u is subharmonic with sublinear growth, then u is constant.*

Proof. We conclude from Theorem 2.7 and the fact that $f(r) > 0$ that

$$f(r) = ar^n + o(r^n) \quad \text{as } r \rightarrow \infty,$$

with an appropriate $a > 0$. This implies that we have for a suitably large $R_0 > 0$,

$$\int_0^r f(s)ds \geq \frac{a}{2}r^{n+1} \quad \forall r \geq R_0,$$

and

$$\left| \frac{f(r)r}{\int_0^r f(s)ds} \right| \leq C \quad \forall r \geq R_0,$$

with an appropriate $C > 0$. Subexponential growth of u implies

$$\frac{|u(x)|}{d(p, x)} \rightarrow 0 \quad \text{as } d(p, x) \rightarrow \infty.$$

Consequently, we can find for every $\epsilon > 0$ an $R_1 > 0$ such that

$$\frac{|u(x)|}{d(p, x)} \leq \epsilon,$$

for all $x \in X$ with $d(p, x) \geq R_1$. Using Theorem 8.1, we conclude that, for all $r \geq \max\{R_0, R_1\}$ and all $v \in S_p X$,

$$\langle \text{grad } u(p), v \rangle \leq \frac{f(r)}{\int_0^r f(s)ds} \epsilon r \leq C\epsilon.$$

Since $\epsilon > 0$ and $v \in S_p X$ were arbitrary, this shows that $\text{grad } u(p) = 0$ for all $p \in X$, i.e., u is a constant function. \square

Remark. *Let (X, g) be a harmonic manifold.*

(a) *If u is a harmonic function, then u^2 is subharmonic:*

$$\Delta u^2 = u \cdot \Delta u + \Delta u \cdot u + 2\langle \text{grad } u, \text{grad } u \rangle = 2\|\text{grad } u\|^2 \geq 0.$$

(b) *A similar argument shows for arbitrary harmonic manifolds (X, g) (also those with exponential volume growth) that if u is subharmonic (i.e., $\Delta u \geq 0$) with linear growth, then $\|\text{grad } u\|$ is a bounded function. Examples of those functions are Busemann functions $b_v(q) = \lim_{t \rightarrow \infty} d(q, c_v(t)) - t$. In this case we have $\Delta b_v(q) = h \geq 0$ (h denotes the mean curvature of the horospheres) and $\|\text{grad } b_v(q)\| = 1$ for all $q \in X$.*

(c) *If all horospheres of X are minimal, i.e., $h = 0$, then all Busemann functions are harmonic functions with linear growth. Moreover, X has polynomial volume growth, by Corollary 2.8. This shows that Corollary 8.2 is sharp.*

9. SPECIAL HARMONIC FUNCTIONS

Based on their eigenfunctions φ_v on geodesic spheres of harmonic manifolds, Ranjan/Shah introduced in [RSh2, formula (4.1)] interesting harmonic functions, denoted by h_v . In the case of harmonic manifolds with subexponential growth, these functions h_v have linear growth. In the case of harmonic manifolds with exponential growth, the functions h_v are bounded. These functions can be used in the latter case to construct a diffeomorphism from the harmonic manifold (X, g) of dimension n to an n -dimensional Euclidean open ball (see Chapter 10). Moreover, this diffeomorphism is a harmonic map (see [RSh2, p. 690, Remarks (i,ii)]).

Theorem 9.1. *Let (X, g) be a harmonic manifold of dimension n and $v \in T_p X$. Then the function*

$$h_v(q) = \mu(d_p(q))\varphi_v(q)$$

with $\mu(r) = \frac{\int_0^r f(s)ds}{f(r)}$ is harmonic.

We have seen earlier in the proof of flatness of noncompact harmonic spaces with $h = 0$ that the function μ played a crucial role.

Proof. Using (7.1), the fact that φ_v is a Laplace eigenfunction on geodesic spheres, and $\text{grad}(\mu \circ d_p) \perp \text{grad} \varphi_v$, we obtain

$$\begin{aligned} \Delta h_v(q) &= \left(\mu''(d_p(q)) + \frac{f'}{f}(d_p(q))\mu'(d_p(q)) \right) \varphi_v(q) \\ &\quad + \mu(d_p(q)) \left(\frac{f'}{f} \right)'(d_p(q)) \varphi_v(q), \end{aligned}$$

i.e.,

$$\begin{aligned} \Delta h_v(q) &= \varphi_v(q) \left(\mu''(d_p(q)) + \left(\frac{f'}{f} \mu \right)'(d_p(q)) \right) \\ &= \varphi_v(q) \left(\mu' + \frac{f'}{f} \mu \right)' \circ d_p(q). \end{aligned}$$

Therefore it suffices to show that $\mu' + \frac{f'}{f} \mu$ is constant. This follows immediately from

$$(9.1) \quad \mu'(r) = \frac{f(r)^2 - \int_0^r f(s)ds f'(r)}{f(r)^2} = 1 - \frac{f'}{f}(r)\mu(r).$$

□

Remark. (a) If (X, g) is a harmonic manifold with subexponential growth, we have $\frac{\log f(r)}{r} \rightarrow 0$. Similar arguments as those in the proof of Corollary 8.2 show that then $\mu(r) = O(r)$, as $r \rightarrow \infty$, i.e., $\frac{\mu(r)}{r}$

is bounded and h_v has linear growth. (We have already seen that a harmonic manifold with subexponential growth must be flat. But we like to stress that the growth behaviour of h_v can be derived without this much stronger result.)

(b) If (X, g) is a harmonic manifold with exponential growth, we have

$$(9.2) \quad \lim_{r \rightarrow \infty} \mu(r) = \lim_{r \rightarrow \infty} \frac{\int_0^r f(s) ds}{f(r)} = \lim_{r \rightarrow \infty} \frac{f(r)}{f'(r)} = \frac{1}{h},$$

where $h > 0$ denotes the mean curvature of the horospheres of X . In this case, h_v is a bounded harmonic function.

We end this chapter with the following straightforward observation.

Lemma 9.2. *Let (X, g) be a harmonic manifold, $u \in C^\infty(X)$ be a harmonic function and $r > 0$ with*

$$(9.3) \quad u|_{S_r(p)} \in \text{span}\{\varphi_v|_{S_r(p)} \mid v \in T_p X\}.$$

Then there exists a vector $v \in T_p X$, such that we have $u = h_v$.

Proof. By the assumption (9.3), we can find constants $\alpha_i \in \mathbb{R}$ such that $u|_{S_r(p)} = \sum_{i=1}^n \mu(r) \alpha_i \varphi_{e_i} = \mu(r) \varphi_{\sum_{i=1}^n \alpha_i e_i}$. Let $v = \sum_{i=1}^n \alpha_i e_i$. Then

$$h_v|_{S_r(p)} - u|_{S_r(p)} = 0,$$

and $h_v - u$ is harmonic. By the maximum principle, $h_v - u$ vanishes on the closed ball $B_r(p)$. By the unique continuation principle (see [Aro]), we conclude that $h_v - u = 0$. \square

10. BALL MODEL OF A NONCOMPACT HARMONIC SPACE

Let (X, g) be a simply connected, noncompact harmonic manifold of dimension n with $h > 0$, $p \in X$ and $v \in T_p X$. Recall from Theorem 9.1 that

$$h_v(q) = \mu(d(p, q)) \cdot \overbrace{\langle v, w_p(q) \rangle}^{\varphi_v(q)},$$

where $w_p(q) \in S_p X$, $q = \exp_p(d(p, q)w_p(q))$ and $\mu(r) = \frac{\int_0^r f(s) ds}{f(r)}$. Moreover, we have $\mu(0) = 0$ and $\mu'(0) = \lim_{r \rightarrow 0} \mu'(r) = 1/n$ (see Proposition 6.1). Since

$$h_v(\exp_p(rw)) = \mu(r) \langle v, w \rangle \quad \forall w \in S_p X, v \in T_p X,$$

we have $h_v(p) = 0$, and h_v is differentiable in p with

$$\begin{aligned} \langle \text{grad } h_v(p), w \rangle &= \left. \frac{d}{dt} \right|_{t=0} h_v(\exp_p(tw)) = \lim_{t \rightarrow 0} \frac{\mu(r)}{r} \langle v, w \rangle \\ &= \mu'(0) \langle v, w \rangle = \frac{1}{n} \langle v, w \rangle, \end{aligned}$$

i.e., $\text{grad } h_v(p) = \frac{v}{n}$. For $q \neq p$, we have

$$\text{grad } h_v(q) = \mu'(d(p, q))\varphi_v(q) \text{grad } d_p(q) + \mu(d(p, q)) \cdot \text{grad } \varphi_v(q).$$

From (7.2) we deduce

$$\text{grad } \varphi_v(q) = A_{w_p(q)}^*(r)^{-1} (v - \langle v, w_p(q) \rangle w_p(q)) \in T_q S_r(p),$$

where $r = d(p, q)$. Moreover, we know from Theorem 9.1 that $\Delta h_v(q) = 0$.

Let \mathcal{O}_p be the set of all orthonormal bases of $T_p X$ and $E_p = (e_1, \dots, e_n) \in \mathcal{O}_p$. Define

$$F_{E_p}(q) = (h_{e_1}(q), \dots, h_{e_n}(q)).$$

Theorem 10.1. *For all $E_p \in \mathcal{O}_p$, the map*

$$F_{E_p} : X \rightarrow B_{\frac{1}{h}}(0) = \{y \in \mathbb{R}^n \mid \|y\| < \frac{1}{h}\}$$

is a harmonic map and a diffeomorphism.

Proof. F_{E_p} is a harmonic map since all its component functions are harmonic functions. The proof that F_{E_p} is a diffeomorphism proceeds in two steps.

Step 1: F_{E_p} is a bijection, i.e., for every $y \in B_{\frac{1}{h}}(0)$ there exists a unique $q \in X$ with $F_{E_p}(q) = y$. We have to solve

$$\begin{aligned} F_{E_p}(q) &= (h_{e_1}(q), \dots, h_{e_n}(q)) = (y_1, \dots, y_n) \\ &= \mu(d(p, q))(\langle e_1, w_p(q) \rangle, \dots, \langle e_n, w_p(q) \rangle) \end{aligned}$$

This implies $\sum_{i=1}^n y_i^2 = \mu^2(d(p, q))$, i.e., $\|y\| = \mu(d(p, q))$, which defines $\mu(d(p, q))$ uniquely since $\mu : [0, \infty) \rightarrow [0, \frac{1}{h})$ is a bijection ($\mu(0) = 0$, $\mu'(r) > 0$ by the Strong Maximum Principle (see the proof of [RSh2, Lemma 4.1]), $\mu(r) \rightarrow \frac{1}{h}$ for $r \rightarrow \infty$ (see (9.2))). Furthermore

$$w_p(q) = \frac{1}{\|y\|} \sum y_i e_i$$

$$\text{and } q = \exp_p \left(\underbrace{(\mu^{-1}(\|y\|))}_{d(p, q)} \underbrace{\frac{1}{\|y\|} \sum y_i e_i}_{\in S_p X} \right).$$

Step 2: $DF_{E(p)}(q)w = 0$ for $w \in T_q X$ implies $w = 0$. The assumption

$$DF_{E(p)}(q)w = (\langle \text{grad } h_{e_1}(q), w \rangle, \dots, \langle \text{grad } h_{e_n}(q), w \rangle) = 0$$

implies that $\langle \text{grad } h_{e_i}(q), w \rangle = 0$ for all i . This, in turn, implies for all $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$:

$$\begin{aligned}
0 = \sum_{i=1}^n \alpha_i \langle \text{grad } h_{e_i}(q), w \rangle &= \left\langle \sum_{i=1}^n \alpha_i \text{grad } h_{e_i}(q), w \right\rangle \\
&= \langle \text{grad } h_{\sum \alpha_i e_i}(q), w \rangle,
\end{aligned}$$

since a straightforward consequence of the definition of h_v is

$$\lambda_1 h_{v_1}(q) + \lambda_2 h_{v_2}(q) = \mu(d(p, q)) \cdot \langle \lambda_1 v_1 + \lambda_2 v_2, w_p(q) \rangle = h_{\lambda_1 v_1 + \lambda_2 v_2}(q).$$

Thus we conclude that $\langle \text{grad } h_v(q), w \rangle = 0$ for all $v \in T_p X$. Using the above formula for the gradient of h_v , we obtain for all $v \in T_p X$:

$$\langle \mu'(d(p, q)) \varphi_v(q) \underbrace{\text{grad } d_p(q)}_{(T_q S_r(p))^\perp} + \mu(d(p, q)) \underbrace{\text{grad } \varphi_v(q)}_{T_q S_r(p)}, w \rangle = 0.$$

Next, we write $w \in T_q X$ as

$$w = \underbrace{\langle w, \text{grad } d_p(q) \rangle \text{grad } d_p(q)}_{(T_q S_r(p))^\perp} + w'$$

with $w' \in T_q S_r(p)$. This implies

$$\begin{aligned}
(10.1) \quad 0 &= \mu'(d(p, q)) \varphi_v(q) \langle w, \text{grad } d_p(q) \rangle \\
&+ \mu(d(p, q)) \langle \text{grad } \varphi_v(q), w' \rangle \quad \forall v \in T_p X.
\end{aligned}$$

Recall that we have

$$\text{grad } \varphi_v(q) = A_{w_p(q)}^*(r)^{-1}(v - \langle v, w_p(q) \rangle w_p(q)).$$

Choose $v = w_p(q)$, i.e. $v - \langle v, w_p(q) \rangle w_p(q) = 0$, i.e. $\text{grad } \varphi_v(q) = 0$. Then

$$0 = \mu'(d(p, q)) \varphi_{w_p(q)}(q) \langle w, \text{grad } d_p(q) \rangle.$$

Since $\varphi_{w_p(q)}(q) = \langle w_p(q), w_p(q) \rangle = 1$ and $\mu'(d(p, q)) > 0$, we obtain

$$(10.2) \quad \langle w, \text{grad } d_p(q) \rangle = 0.$$

This implies together with (10.1) that

$$(10.3) \quad 0 = \mu(d(p, q)) \langle \text{grad } \varphi_v(q), w' \rangle \quad \forall v \in T_p X.$$

For $v \in (w_p(q))^\perp$ we have $\text{grad } \varphi_v(q) = (A_{w_p(q)}^*(r))^{-1}(v)$, and since $(A_{w_p(q)}^*(r))^{-1} : (w_p(q))^\perp \rightarrow T_q S_r(p)$ is an isomorphism, we can realise every vector in $T_q S_r(p)$ as $\text{grad } \varphi_v(q)$, in particular we can find $v \in T_p X$ such that $w' = \text{grad } \varphi_v(q)$. Putting this into (10.3) yields

$$0 = \underbrace{\mu(d(p, q))}_{>0} \langle w', w' \rangle,$$

i.e., $w' = 0$. Equation (10.2) and $w' = 0$ imply that

$$w = \underbrace{\langle w, \text{grad } d_p(q) \rangle \text{grad } d_p(q)}_{=0} + w' = 0,$$

finishing the proof. \square

REMARK The above proof can be simplified by using the Cartan-Hadamard Theorem: Since X is simply connected and has no conjugate points, the map $\exp_p : T_p X \rightarrow X$ is a diffeomorphism. We can view F_{E_p} as a radial rescaling of the inverse exponential map, followed by a canonical identification of $T_p X$ with \mathbb{R}^n via the basis e_1, \dots, e_n .

Next, we calculate F_{E_p} in the case of the hyperbolic plane. We realise the hyperbolic plane as the Poincaré unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ (see Figure 2) with metric

$$g = \frac{4(dx^2 + dy^2)}{(1 - |z|^2)^2}.$$

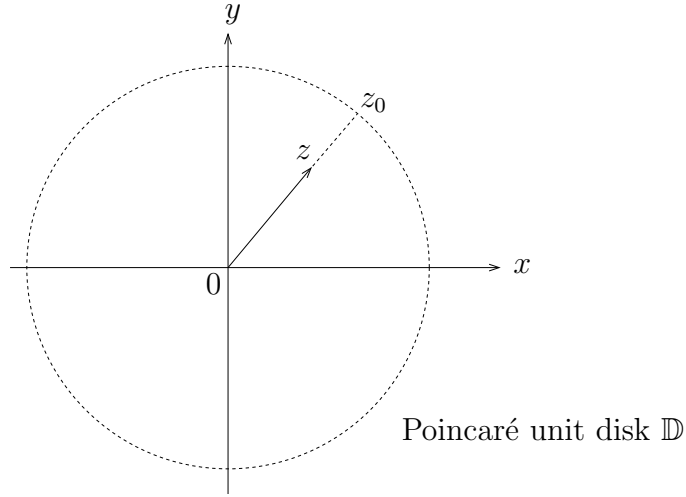


FIGURE 2. The Poincaré unit disk model of the hyperbolic plane

Let $z \in \mathbb{D}$ and $|z| = r < 1$. Let $c : [0, r] \rightarrow \mathbb{D}$ be the curve $c(t) = t \cdot z_0$ with $z_0 = \frac{z}{|z|} \in S^1$. Then

$$\|c'(t)\|_{\mathbb{D}}^2 = \frac{4}{1 - t^2}$$

and

$$\begin{aligned} d_{\mathbb{D}}(z, 0) &= \int_0^r \|c'(t)\|_{\mathbb{D}} dt = 2 \int_0^r \frac{1}{1 - t^2} dt = 2 \int_0^{\operatorname{arctanh}(r)} \cosh^2 u (1 - \tanh^2 u) du \\ &= 2 \int_0^{\operatorname{arctanh}(r)} du = 2 \operatorname{arctanh}(r), \end{aligned}$$

using the substitution $t = \tanh(u)$, which implies $dt = (1 - \tanh^2 u)du$ and $1 - t^2 = 1 - \tanh^2 u = \frac{1}{\cosh^2 u}$. Note that $f(t) = \sinh(t)$, i.e.,

$$\begin{aligned}\mu(r) &= \frac{\int_0^r f(t)dt}{f(r)} \\ &= \frac{\cosh r - 1}{\sinh r}.\end{aligned}$$

Let $p = 0 \in \mathbb{D}$ and $E_p = \{e_1, e_2\}$, where e_i is the standard basis in $\mathbb{R}^2 \cong \mathbb{C}$. Note that $h = 1$. Then

$$F = F_{E_p} : \mathbb{D} \rightarrow B_{\frac{1}{h}}(0) = B_1(0)$$

is given by $F(z) = \mu(d_{\mathbb{D}}(0, z)) \cdot \frac{z}{|z|}$. Since $d_{\mathbb{D}}(0, z) = 2\operatorname{arctanh}(r)$ for $z \in \mathbb{D}$ with $|z| = r$, we obtain

$$\begin{aligned}\mu(d_{\mathbb{D}}(0, z)) &= \frac{\cosh(2\operatorname{arctanh}(r)) - 1}{\sinh(2\operatorname{arctanh}(r))} \\ &= \frac{\cosh^2(\operatorname{arctanh}(r)) + \sinh^2(\operatorname{arctanh}(r)) - 1}{2\sinh(\operatorname{arctanh}(r))\cosh(\operatorname{arctanh}(r))} \\ &= \frac{\sinh^2(\operatorname{arctanh}(r))}{\sinh(\operatorname{arctanh}(r))\cosh(\operatorname{arctanh}(r))} = r.\end{aligned}$$

This implies that

$$F(z) = r \cdot \frac{z}{r} = z,$$

i.e., the model of the hyperbolic plane obtained via the harmonic map F is the Poincaré unit disk, as we would have expected.

11. THE BUSEMANN BOUNDARY

Let (X, g) be a noncompact complete Riemannian manifold and $p \in X$ fixed. Define $B^p : X \rightarrow C(X)$, where $B^p(y) = B_y^p : X \rightarrow \mathbb{R}$ is given by

$$B_y^p(x) = d(y, x) - d(p, y).$$

Lemma 11.1. *Assume that $C(X)$ carries the topology of uniform convergence on compact sets. Then $B^p : X \rightarrow C(X)$ is injective and continuous. Furthermore, $\overline{B^p(X)}$ is compact in $C(X)$.*

Proof. Consider $y, y' \in X$ such that $B_y^p = B_{y'}^p$. This implies

$$d(x, y') - d(y', p) = d(x, y) - d(y, p)$$

for all $x \in X$. In particular, for $x = y$ and $x = y'$ we obtain

$$d(y, y') = d(y', p) - d(y, p) \text{ and } -d(y', p) = d(y, y') - d(y, p),$$

and therefore $d(y, y') = 0$, which shows injectivity.

To show that B^p is continuous, we have to prove that, for each sequence y_n converging to y , the sequence $B_{y_n}^p$ converges uniformly on

compact subsets to B_y^p . The continuity of d implies that $B_{y_n}^p$ converges pointwise to B_y . Since

$$|B_y^p(x) - B_y^p(x')| \leq d(x, x')$$

the family $B_{y_n}^p$ is equicontinuous and by Arzela-Ascoli converges uniformly on compact subsets. Since $\overline{B^p(X)}$ is a Fréchet space, it is metrizable and we only have to check sequential compactness. Let $B_{y_n}^p$ be an arbitrary sequence. Since $|B_y^p(x)| \leq d(x, p)$ is bounded on compact subsets independent of y , Arzela-Ascoli implies the existence of a subsequence converging uniformly on compact sets to a continuous function f . \square

Definition 11.2. *The set $\overline{B^p(X)}$ is called the Busemann compactification and $\partial_B^p X := \overline{B^p(X)} \setminus B^p(X)$ is called the Busemann boundary with respect to p . Using the bijection $B^p : X \rightarrow B^p(X)$, a sequence $y_n \in X$ converges to $\xi \in \partial_B^p X$ (in the Busemann topology) iff $B_{y_n}^p$ converges to ξ uniformly on compact subsets.*

Note that the point p is only a normalization (i.e., all functions $f \in \overline{B^p(X)}$ satisfy $f(p) = 0$), and the convergence of a sequence is independent of the choice of p . More precisely, if p' is another point, we have

$$B_y^p(x) - B_y^{p'}(x) = d(y, p') - d(y, p) = B_y^p(p').$$

Hence, for a given sequence $y_n \in X$, the functions $B_{y_n}^p$ converge uniformly on compact sets to ξ if and only if the functions $B_{y_n}^{p'} = B_{y_n}^p - B_{y_n}^p(p')$ converge uniformly on compact sets to $\xi - \xi(p')$.

Note also that $y_n \rightarrow \xi \in \partial_B^p X$ means necessarily that $d(p, y_n) \rightarrow \infty$: If y_n would have a subsequence y_{n_j} with $d(p, y_{n_j}) \leq C$ for some $C > 0$, then a subsequence of y_{n_j} would be convergent to a point $y \in X$. Uniqueness of the limit would imply $B_{y_n}^p \rightarrow B_y^p$, uniformly on compact sets, but $B_y^p \notin \partial_B^p X$.

If (X, g) has no conjugate points, the Busemann boundary $\partial_B^p X$ can be represented via Busemann functions. Let S_v be the stable Jacobi tensor along c_v as defined in Lemma 3.1. The corresponding unstable Jacobi tensor U_v along c_v is given by $U_v(t) = S_{-v}(-t)$. We say that (X, g) has *continuous asymptote* if the stable solution $v \mapsto S'_v(0)$ of the Ricatti equation is continuous. Let us mention some important results for manifolds without conjugate points (see [Kn2, Satz 3.5] and [Zi3, Lemma 4.3]).

Proposition 11.3. *Let (X, g) be a complete, simply connected Riemannian manifold without conjugate points. For $v \in SX$ consider $b_{v,t} : X \rightarrow \mathbb{R}$, defined by $b_{v,t}(x) = d(x, c_v(t)) - t$. Then $b_v(x) = \lim_{t \rightarrow \infty} b_{v,t}(x)$ exist and defines a C^1 function on X . The functions $\text{grad } b_{v,t}$ converge to $\text{grad } b_v$ in $C(X)$, i.e., the convergence is uniformly on compact sets. Furthermore, $\text{grad } b_v$ is Lipschitz continuous.*

If (X, g) has continuous asymptote, then the map $b : SX \rightarrow C(X)$ with $v \mapsto b_v$ is continuous.

REMARK For a proof of the first two properties in Proposition 11.3 above under the additional assumption of a continuous asymptote see [Es, Prop. 1 and 2]. Note that noncompact harmonic manifolds have continuous asymptote (see Ranjan and Shah [RSh3] or Zimmer [Zi3, Lemma 5.4]).

The following proposition is due to Zimmer [Zi3, Prop. 2.11 and Lemma 4.5]. For convenience we provide a proof.

Proposition 11.4. *Assume that (X, g) is a complete, simply connected manifold without conjugate points such that the map $b : SX \rightarrow C(X)$ is continuous. Then the following properties hold.*

- (1) *The sequence $b_{v,t}(x)$ converges uniformly on compact subsets of $X \times SX$ to $b_v(x)$.*
- (2) *Let $\partial_B^{p_0} X$ be the Busemann boundary of X with respect to $p_0 \in X$. Then, for any $p \in X$ the map $\varphi_p^{p_0} : S_p X \rightarrow \partial_B^{p_0} X$ with $\varphi_p^{p_0}(v) := b_v - b_v(p_0)$ is a homeomorphism.*
- (3) *A sequence $y_n = \exp_p(t_n v_n) \in X$ with $v_n \in S_p X$ and $t_n \geq 0$ converges to a point $\xi \in \partial_B^{p_0} X$ if and only if $t_n \rightarrow \infty$ and there exists $v \in S_p X$ with $v_n \rightarrow v$. In particular, ξ is given by $b_v - b_v(p_0)$.*

Proof. (1) From the triangle inequality, we obtain for all $v \in SX$ and $x \in X$ that $b_{v,t}(x) \leq b_{v,s}(x)$ if $s \leq t$. Since the map $(x, v) \rightarrow b_v(x)$ is continuous, Dini's theorem implies that $b_{v,t}(x)$ converges uniformly on compact subsets of $X \times SX$ to $b_v(x)$.

(2) Proposition 11.3 implies that $\varphi_p^{p_0}$ is continuous. To show surjectivity assume that $B_{y_n}^{p_0}$ converges to $\xi \in \partial_B^{p_0} X$. Define $v_n \in S_p X$ and $t_n \geq 0$ such that $c_{v_n}(t_n) = y_n$. Then

$$B_{y_n}^{p_0}(x) = d(x, c_{v_n}(t_n)) - d(c_{v_n}(t_n), p_0) = b_{v_n, t_n}(x) - b_{v_n, t_n}(p_0).$$

By passing to a subsequence if necessary, we can assume that v_n converges to $v \in S_p X$. Recall that $t_n \rightarrow \infty$. Because of (1), the right hand converges to $\xi = b_v - b_v(p_0)$. This shows that $\varphi_p^{p_0}$ is surjective. The map is also injective since $\varphi_p^{p_0}(v) = \varphi_p^{p_0}(w)$ implies $-v = \text{grad } b_v(p) = \text{grad } b_w(p) = -w$. Therefore, $\varphi_p^{p_0} : S_p X \rightarrow \partial_B^{p_0} X$ is continuous and bijective and, since $S_p X$ is compact, $\varphi_p^{p_0}$ is a homeomorphism.

(3) Let $y_n = \exp_p(t_n v_n) \in X$ with $v_n \in S_p X$ and $t_n \geq 0$ be a sequence. Assume that y_n converges to $\xi \in \partial_B^{p_0} X$, i.e., $B_{y_n}^{p_0}$ converges to $\xi = b_v - b_v(p_0)$ for some $v \in S_p X$. Since $d(y_n, p) \rightarrow \infty$, we know that $t_n \rightarrow \infty$. As above we have

$$B_{y_n}^{p_0}(x) = d(x, c_{v_n}(t_n)) - d(c_{v_n}(t_n), p_0) = b_{v_n, t_n}(x) - b_{v_n, t_n}(p_0).$$

If $v_n \not\rightarrow v$, we would have a subsequence of v_n converging to $w \in S_p X$ with $w \neq v$. But then, following the arguments in (2), this subsequence

would converge to $b_w - b_w(p_0)$, violating the injectivity of the map $\varphi_p^{p_0}$. Conversely, if $v_n \rightarrow v$ and $t_n \rightarrow \infty$, (1) implies $B_{y_n}^{p_0} = b_{v_n, t_n} - b_{v_n, t_n}(p_0) \rightarrow b_v - b_v(p_0) =: \xi$, uniformly on compact subsets of X . \square

12. VISIBILITY MEASURES AND THEIR RADON-NYKODYM DERIVATIVE

Let (X, g) be a noncompact, simply connected harmonic manifold of dimension n . We choose a reference point $p_0 \in X$ and define $\partial_B X := \partial_B^{p_0} X$. For any other point $p \in X$, we know from Proposition 11.4 that the map $\varphi_p^{p_0} \circ (\varphi_p^p)^{-1} : \partial_B X \rightarrow \partial_B^p X$ is a homeomorphism, identifying both Busemann boundaries in a canonical way. Moreover, the homeomorphisms $\varphi_p^{p_0} : S_p X \rightarrow \partial_B^{p_0} X$ motivate the following definition.

Definition 12.1. Let $\mathcal{M}_1(\partial_B X)$ denote the space of Borel probability measures on the Busemann boundary $\partial_B X$. For every $p \in X$, we define $\mu_p \in \mathcal{M}_1(\partial_B X)$ via

$$\int_{\partial_B X} f(\xi) d\mu_p(\xi) = \frac{1}{\omega_n} \int_{S_p X} f(\varphi_p^{p_0}(v)) d\theta_p(v) \quad \forall f \in C(\partial_B X),$$

where ω_n is the volume of the $(n-1)$ -dimensional standard unit sphere and $d\theta_p$ is the volume element of $S_p X$ induced by the Riemannian metric.

We call μ_p the visibility measure of (X, g) at the point p .

We will see that any two visibility measures $\mu_p, \mu_q \in \mathcal{M}_1(\partial_B X)$ are absolutely continuous, by calculating their Radon-Nykodym derivative via a limiting process. This needs some preparations.

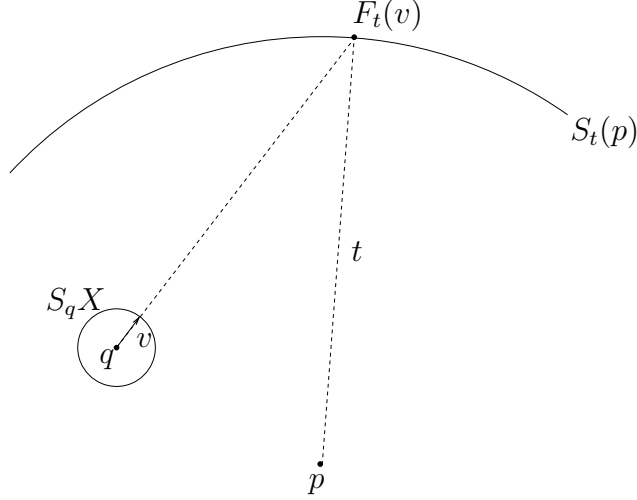
Lemma 12.2. Let (X, g) be a noncompact simply connected harmonic space. For all $p, q \in X$ there exists a $t(p, q) > 0$ such that for all $t \geq t(p, q)$ and all $v \in S_q X$ the geodesic ray $c_v : [0, \infty) \rightarrow X$ intersects $S_t(p)$ in a unique point $F_t(v)$ (see Figure 3). In particular, the map $F_t : S_q X \rightarrow S_t(p)$ is bijective.

Proof. Let $a(t)$ be as in Corollary 3.4. Choose t_0 such that for all $t \geq t_0$ we have $2d(p, q) \leq a(t)$. Define

$$t(p, q) = \max\{d(p, q) + 1, t_0\}.$$

In particular, q lies in the ball of radius t around p , for all $t \geq t(p, q)$, and hence for all $v \in S_q X$ the geodesic ray $c_v : [0, \infty) \rightarrow X$ intersects $S_t(p)$. Let $t \geq t(p, q)$, and assume that $q' = c_v(t_1)$ is the second intersection point. Let $w \in S_p X$ be the unique vector such that $c_w(t) = q'$. Since $\dot{c}_v(t_1)$ either points into $B_t(p)$ or is tangent to $S_t(p)$ we have

$$\angle(\dot{c}_v(t_1), \dot{c}_w(t)) \geq \pi/2.$$

FIGURE 3. Illustration of the map $F_t : S_q X \rightarrow S_t(p)$

Using the triangle inequality we obtain

$$t - d(p, q) \leq t_1 \leq t + d(p, q)$$

Using Corollary 3.4, we obtain for all $s \geq 0$

$$d(c_v(t_1 - s), c_w(t - s)) \geq a(s)\pi/2.$$

In particular for $s = t$ this yields

$$a(t)\pi/2 \leq d(c_v(t_1 - t), p) \leq d(c_v(t_1 - t), q) + d(q, p) \leq 2d(p, q) \leq a(t),$$

which is a contradiction. Hence, a second intersection point cannot occur. \square

Proposition 12.3. *Let (X, g) be a complete, simply connected non-compact manifold without conjugate points and $p, q \in X$. Consider the map $F_t : S_q X \rightarrow S_t(p)$, where $F_t(v)$ is the first intersection point of the geodesic ray $c_v : [0, \infty) \rightarrow X$ with $S_t(p)$. If q is contained in the ball of radius t about p , this map is well defined. Then the Jacobian of F_t is given by*

$$\text{Jac } F_t(v) = \frac{\det A_v(d(q, F_t(v)))}{\langle N_p(F_t(v)), N_q(F_t(v)) \rangle},$$

where $N_x(y) = (\text{grad } d_x)(y)$ and A_v is Jacobitensor along c_v with $A_v(0) = 0$ and $A'_v(0) = \text{id}$. Note that we have $\det A_v(s) = f(s)$ if (X, g) is harmonic.

Proof. Choose a curve $\gamma : (-\epsilon, \epsilon) \rightarrow S_q X$ with $\gamma(0) = v \in S_q X$. Then

$$F_t(\gamma(s)) = \exp_q(d(q, F_t(\gamma(s))) \cdot \gamma(s)),$$

and, using the chain rule and the product rule,

$$DF_t(v)(\dot{\gamma}(0)) = D \exp_q(d(q, F_t(v)) \cdot v)(\langle N_q(F_t(v)), DF_t(v)\dot{\gamma}(0) \rangle v + d(q, F_t(v)) \cdot \dot{\gamma}(0)).$$

Note that $\dot{\gamma}(0) \perp v$.

We have

$$D \exp_q(tv)(tw) = Y(t)(w) = J(t),$$

where Y is the Jacobi tensor along c_v with $Y(0) = 0$ and $Y'(0) = \text{id}$, and therefore J is a Jacobi field along c satisfying $J(0) = 0$ and $J'(0) = w$. Note that Y and A_v are related by $A_v = Y|_{(c'_v)^\perp}$. In particular, we have

$$D \exp_q(tv)(tv) = tD \exp_q(tv)(v) = tc'_v(t).$$

This yields

$$\begin{aligned} DF_t(v)(\dot{\gamma}(0)) &= \langle N_q(F_t(v)), DF_t(v)\dot{\gamma}(0) \rangle D \exp_q(d(q, F_t(v))v)(v) \\ &\quad + A_v(d(q, F_t(v)))(\dot{\gamma}(0)) \\ &= \langle N_q(F_t(v)), DF_t(v)\dot{\gamma}(0) \rangle c'_v(d(q, F_t(v))) \\ &\quad + A_v(d(q, F_t(v)))(\dot{\gamma}(0)). \end{aligned}$$

Consequently,

$$DF_t(v)(\dot{\gamma}(0)) = \langle N_q(F_t(v)), DF_t(v)\dot{\gamma}(0) \rangle N_q(F_t(v)) + A_v(d(q, F_t(v)))(\dot{\gamma}(0)).$$

Next, we introduce the map

$$\begin{aligned} L_x : N_p(x)^\perp &\rightarrow N_q(x)^\perp, \\ L_x(w) &= w - \langle w, N_q(x) \rangle N_q(x). \end{aligned}$$

Then we have

$$L_{F_t(v)}(DF_t(v)(\dot{\gamma}(0))) = A_v(d(q, F_t(v)))(\dot{\gamma}(0)).$$

To finish the proof of the above Proposition, we need the following lemma.

Lemma 12.4. $\text{Jac } L_x = |\langle N_p(x), N_q(x) \rangle|$.

Proof. Consider

$$N_p(x)^\perp \cap N_q(x)^\perp = \{w \in T_x X \mid \langle w, N_p(x) \rangle = 0 \text{ and } \langle w, N_q(x) \rangle = 0\}.$$

Then $N_p(x)^\perp \cap N_q(x)^\perp$ has co-dimension one in $N_p(x)^\perp$ and L_x is the identity on $N_p(x)^\perp \cap N_q(x)^\perp$. Let

$$w_0 = N_q(x) - \langle N_q(x), N_p(x) \rangle N_p(x) \in N_p(x)^\perp.$$

The vector w_0 is orthogonal to $N_p(x)^\perp \cap N_q(x)^\perp$ since for all $w \in N_p(x)^\perp \cap N_q(x)^\perp$ we have $\langle w, N_p(x) \rangle = 0$ and $\langle w, N_q(x) \rangle = 0$, and therefore

$$\langle w, w_0 \rangle = \underbrace{\langle w, N_q(x) \rangle}_{=0} - \langle N_q(x), N_p(x) \rangle \underbrace{\langle w, N_p(x) \rangle}_{=0} = 0.$$

Moreover, $L_x w_0$ is also orthogonal to $N_p(x)^\perp \cap N_q(x)^\perp$:

$$\begin{aligned} L_x w_0 &= w_0 - \langle w_0, N_q(x) \rangle N_q(x) \\ &= N_q(x) - \langle N_q(x), N_p(x) \rangle N_p(x) - \langle N_q(x), N_q(x) \rangle N_q(x) \\ &\quad + \langle N_q(x), N_p(x) \rangle^2 N_q(x) \\ &= \langle N_p(x), N_q(x) \rangle (\langle N_p(x), N_q(x) \rangle N_q(x) - N_p(x)), \end{aligned}$$

and consequently $\langle w, L_x w_0 \rangle = 0$ for all w satisfying $\langle w, N_p(x) \rangle = \langle w, N_q(x) \rangle = 0$. Consequently:

$$\text{Jac } L_x = \frac{\|L_x w_0\|}{\|w_0\|}.$$

Since

$$\begin{aligned} \|L_x w_0\|^2 &= \langle N_p(x), N_q(x) \rangle^2 (\langle N_p(x), N_q(x) \rangle^2 + 1 - 2\langle N_p(x), N_q(x) \rangle^2) \\ &= \langle N_p(x), N_q(x) \rangle^2 (1 - \langle N_p(x), N_q(x) \rangle^2) \end{aligned}$$

and

$$\begin{aligned} \|w_0\|^2 &= 1 + \langle N_p(x), N_q(x) \rangle^2 - 2\langle N_p(x), N_q(x) \rangle^2 \\ &= 1 - \langle N_p(x), N_q(x) \rangle^2, \end{aligned}$$

we obtain

$$\begin{aligned} \text{Jac } L_x &= \left(\frac{\langle N_p(x), N_q(x) \rangle^2 (1 - \langle N_p(x), N_q(x) \rangle^2)}{1 - \langle N_p(x), N_q(x) \rangle^2} \right)^{1/2} \\ &= |\langle N_p(x), N_q(x) \rangle|, \end{aligned}$$

which yields the lemma. \square

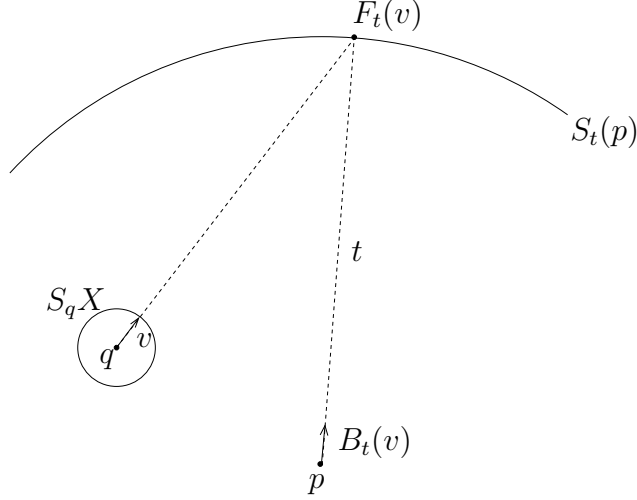
Finally, $L_{F_t(v)} \circ DF_t(v) = A_v(d(q, F_t(v)))$ implies that

$$\text{Jac } F_t(v) = \frac{\det A_v(d(q, F_t(v)))}{\text{Jac } L_{F_t(v)}} = \frac{\det A_v(d(q, F_t(v)))}{\langle N_p(F_t(v)), N_q(F_t(v)) \rangle},$$

finishing the proof of the proposition. \square

Corollary 12.5. *Let (X, g) be a noncompact, simply connected harmonic space. Let $B_t : S_q X \rightarrow S_p X, v \mapsto \frac{1}{t} \exp_p^{-1} \circ F_t(v)$ (see Figure 4). Then we have*

$$\text{Jac } B_t(v) = \frac{f(d(q, F_t(v)))}{f(t)} \cdot \frac{1}{\langle N_p(F_t(v)), N_q(F_t(v)) \rangle}.$$

FIGURE 4. Illustration of the map $B_t : S_q X \rightarrow S_p X$

Proof. Let $u \in S_p X$. Then $D \exp_p(tu) : u^\perp \rightarrow T_{\exp_p(tu)} S_t(p)$ is given by $D \exp_p(tu)(w) = \frac{1}{t} A_u(t)(w)$, and therefore with $u = B_t(v)$,

$$\begin{aligned} \text{Jac } B_t(v) &= \frac{1}{\det A_u(t)} \cdot \text{Jac } F_t(v) \\ &= \frac{\det A_v(d(q, F_t(v)))}{\det A_u(t)} \cdot \frac{1}{\langle N_p(F_t(v)), N_q(F_t(v)) \rangle}. \end{aligned}$$

Since X is harmonic, we have $\det A_v(s) = f(s)$, which finishes the proof of the corollary. \square

Let $f \in C(\partial_B X)$. We know from Lemma 12.2 that $B_t : S_q X \rightarrow S_p X$ is a bijection, for $t > 0$ large enough. Then we have with $f_1 = f \circ \varphi_p^{p_0}$:

$$\begin{aligned} \int_{\partial_B X} f(\xi) d\mu_p(\xi) &= \frac{1}{\omega_n} \int_{S_p X} f_1(w) d\theta_p(w) \\ &= \frac{1}{\omega_n} \int_{S_q X} (f_1 \circ B_t)(v) (\text{Jac } B_t)(v) d\theta_q(v). \end{aligned}$$

We will show that

- (i) $\lim_{t \rightarrow \infty} B_t = (\varphi_p^{p_0})^{-1} \circ (\varphi_q^{p_0})$,
- (ii) There exist constants $t_0 > 0$ and $C > 0$ such that

$$|\text{Jac } B_t(v)| \leq C \quad \forall v \in S_q X, t \geq t_0.$$

- (iii) We have, for all $v \in S_q X$,

$$\lim_{t \rightarrow \infty} \text{Jac } B_t(v) = e^{-hb_v(p)}.$$

Having these facts, we conclude with Lebesgue's dominated convergence that

$$\begin{aligned} \int_{\partial_B X} f(\xi) d\mu_p(\xi) &= \lim_{t \rightarrow \infty} \frac{1}{\omega_n} \int_{S_q X} (f_1 \circ B_t)(v) (\text{Jac } B_t)(v) d\theta_q(v) \\ &= \frac{1}{\omega_n} \int_{S_q X} (f \circ \varphi_q^{p_0}) e^{-hb_v(p)} d\theta_q(v) \\ &= \int_{\partial_B X} f(\xi) e^{-hb_{q,\xi}(p)} d\mu_q(\xi), \end{aligned}$$

with $b_{q,\xi} = \xi - \xi(q)$ for $q \in X$ and $\xi \in \partial_B X$. This shows the following fact:

Theorem 12.6. *Let (X, g) be a simply connected noncompact harmonic manifold with reference point $p_0 \in X$. Let $(\mu_p)_{p \in X}$ be the associated family of visibility measures. Then these measures are pairwise absolutely continuous and we have*

$$\frac{d\mu_p}{d\mu_q}(\xi) = e^{-hb_{q,\xi}(p)}.$$

It remains to prove (i), (ii) and (iii) above.

Proof of (i): Let $t_n \rightarrow \infty$ and $s_n \geq 0$, $w_n = B_{t_n}(v) \in S_p X$ such that $y_n = \exp_q(s_n v) = \exp_p(t_n w_n)$. We obviously have $s_n \rightarrow \infty$ and $y_n \rightarrow b_v - b_v(q)$. Let w_{n_j} be a convergent subsequence of $w_n = B_{t_n}(v)$ with limit $w \in S_p X$. Then we have $y_{n_j} \rightarrow b_w - b_w(p)$, by Proposition 11.4(3), and

$$\varphi_p^{p_0}(v) = b_v - b_v(q) = b_w - b_w(p) = \varphi_q^{p_0}(w).$$

This shows that $\lim_{n \rightarrow \infty} B_{t_n}(v) = (\varphi_q^{p_0})^{-1} \circ \varphi_p^{p_0}(v)$. \square

For the proof of (ii), we need the following lemma:

Lemma 12.7. *For every $\epsilon > 0$, there exists $t_0 > 0$ such that we have for all $v \in S_q X$*

$$|\langle N_p(F_t(v)), N_q(F_t(v)) \rangle - 1| < \epsilon \quad \forall t \geq t_0.$$

Proof. This is an easy consequence of corollary 3.4. \square

Proof of (ii): Since $f(t) > 0$ is an exponential polynomial, there exists $m > 0$ such that

$$(12.1) \quad \frac{f(t)}{t^m e^{ht}} \rightarrow 1 \quad \text{as } t \rightarrow \infty.$$

Therefore, there exists $t_0 > 0$ such that

$$\frac{1}{2} t^m e^{ht} \leq f(t) \leq \frac{3}{2} t^m e^{ht}$$

for all $t \geq t_0$. Using Lemma 12.7 and increasing $t_0 > 0$ if necessary, we can also assume that

$$\langle N_p(F_t(v)), N_q(F_t(v)) \rangle \geq \frac{1}{2}$$

for all $t \geq t_0$. Since $d(q, F_t(v)) \leq t + d(p, q)$, we conclude from Corollary 12.5 for all $t \geq t_0$ and all $v \in S_q X$,

$$|\text{Jac } B_t(v)| \leq 6 \frac{(t + d(p, q))^m e^{h(t+d(p,q))}}{t^m e^{ht}} \leq 6 \left(1 + \frac{d(p, q)}{t_0}\right)^m e^{hd(p,q)}. \quad \square$$

Proof of (iii): This is an immediate consequence of Lemma 12.7 and the following Lemma:

Lemma 12.8. *Using the notation above we have that*

$$\lim_{t \rightarrow \infty} \frac{f(d(q, F_t(v)))}{f(t)} = e^{-hb_v(p)}$$

Proof. Note that $B_{F_t v}^q(x) = d(x, F_t(v)) - d(q, F_t(v))$ converges uniformly to b_v . Hence, for all $\epsilon > 0$ there exists $t_0 > 0$ such that for all $t \geq t_0$ we have

$$|b_v(p) - \underbrace{d(p, F_t(v))}_{=t} + d(q, F_t(v))| < \epsilon.$$

This implies

$$-\epsilon + t - b_v(p) \leq d(q, F_t(v)) \leq \epsilon + t - b_v(p).$$

Note that $f(r)$ is monotone, since $\frac{f'}{f}(r) \geq h \geq 0$. Therefore,

$$f(-\epsilon + t - b_v(p)) \leq f(d(q, F_t(v))) \leq f(\epsilon + t - b_v(p))$$

and

$$\frac{f(-\epsilon + t - b_v(p))}{f(t)} \leq \frac{f(d(q, F_t(v)))}{f(t)} \leq \frac{f(\epsilon + t - b_v(p))}{f(t)}$$

Using (12.1), we obtain for $a \in \mathbb{R}$.

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{f(t-a)}{f(t)} &= \lim_{t \rightarrow \infty} \frac{f(t-a)}{(t-a)^m e^{h(t-a)}} \cdot \frac{(t-a)^m e^{h(t-a)}}{t^m e^{ht}} \cdot \frac{t^m e^{ht}}{f(t)} \\ &= \lim_{t \rightarrow \infty} \left(\frac{t-a}{t}\right)^m e^{-ha} = e^{-ha}. \end{aligned}$$

Hence, for all $\epsilon > 0$, we have

$$e^{-h(b_v(p)+\epsilon)} \leq \liminf_{t \rightarrow \infty} \frac{f(d(q, F_t(v)))}{f(t)} \leq \limsup_{t \rightarrow \infty} \frac{f(d(q, F_t(v)))}{f(t)} \leq e^{-h(b_v(p)-\epsilon)}.$$

This implies the claim. \square

13. THE GREEN'S KERNEL AND THE MARTIN BOUNDARY

Let (X, g) be a simply connected noncompact harmonic manifold. In this chapter we calculate explicitly the Green's kernel, which is the smallest non-negative fundamental solution of the Laplace equation on X , i.e., G is a function defined on $X \times X \setminus \{(x, x) \mid x \in X\}$, and having the following properties:

- (a) $\Delta_x G(x, y) = 0 \quad \forall x \neq y$,
- (b) $G(x, y) \geq 0$ for all $x \neq y$,
- (c) For all $y \in X$, we have $\inf_{x \in X, x \neq y} G(x, y) = 0$,
- (d) $\int_X G(x, y) \Delta \varphi(y) dy = -\varphi(x) \quad \forall \varphi \in C_0^\infty(X)$.

G can also be expressed with the help of the smallest positive fundamental solution of the heat equation $p_t(x, y)$ as

$$G(x, y) = \int_0^\infty p_t(x, y) dt.$$

This implies that $G(x, y) = G(y, x)$. For $\dim X \geq 3$, the Green's kernel has a singularity at the diagonal with the asymptotic

$$G(x, y) = \frac{c_n}{d(x, y)^{n-2}}(1 + o(1)) \quad \text{as } x \rightarrow y,$$

with a fixed constant $c_n > 0$ depending on the dimension.

Since the heat kernel $p_t(x, y)$ of a harmonic space X only depends on $d(x, y)$ (see [Sz, Thm 1.1]), the same holds true for the Green's kernel, i.e., there exists a function \tilde{G} such that $G(x, y) = \tilde{G}(d(x, y))$.

Property (a) means that for $r = d(y, x) = d_y(x)$ we have

$$0 = \Delta_x(\tilde{G} \circ d_y)(x) = \tilde{G}''(r) + \frac{f'}{f}(r)\tilde{G}'(r),$$

i.e.

$$(f\tilde{G}'')(r) + (f'\tilde{G}')(r) = 0,$$

i.e., $(f\tilde{G}')'(r) = 0$. Integration yields $f\tilde{G}' + \beta = 0$. We choose $\beta = \frac{1}{\omega_n} = \frac{1}{\text{vol}(S_y X)}$ and obtain $\tilde{G}'(r) = -\frac{\beta}{f(r)}$.

This leads us to consider $\tilde{G}(r) = \beta \int_r^\infty \frac{dt}{f(t)}$. The above derivations show that

$$G : (X \times X) \setminus \{(x, x) \mid x \in X\} \rightarrow \mathbb{R},$$

defined by $G(x, y) = \tilde{G}(d(x, y))$, satisfies $0 = \Delta_x G(x, y)$ for all $x \neq y$, which shows (a) for this choice of G .

The properties (b) und (c) for this choice of $G(x, y)$ are easily verified.

The following calculation shows (d). We have for all $\varphi \in C_0^\infty(X)$:

$$\begin{aligned}
\langle G(x, \cdot), \Delta\varphi \rangle &= \\
&= \int_X G(x, y) \Delta\varphi(y) dy = \int_0^\infty \int_{S_x X} f(r) \tilde{G}(r) (\Delta\varphi)(c_v(r)) d\theta_x(v) dr \\
&= \int_0^\infty f(r) \tilde{G}(r) \int_{S_x X} (\varphi \circ c_v)''(r) + \frac{f'}{f}(r) (\varphi \circ c_v)'(r) d\theta_x(v) dr \\
&= \beta \int_0^\infty f(r) \int_r^\infty \frac{1}{f(t)} \int_{S_x X} (\varphi \circ c_v)''(r) + \frac{f'}{f}(r) (\varphi \circ c_v)'(r) d\theta_x(v) dt dr \\
&= \beta \int_{S_x X} \int_0^\infty \frac{1}{f(t)} \int_0^t \underbrace{f(r) (\varphi \circ c_v)''(r) + f'(r) (\varphi \circ c_v)'(r)}_{(f \cdot (\varphi \circ c_v)')'(r)} dr dt d\theta_x(v) \\
&= \beta \int_{S_x X} \int_0^\infty \frac{1}{f(t)} f(t) (\varphi \circ c_v)'(t) dt d\theta_x(v) \\
&= -\beta \int_{S_x X} (\varphi \circ c_v)(0) d\theta_x(v) = -\varphi(x) \cdot \beta \cdot \text{vol}(S_x X) = -\varphi(x).
\end{aligned}$$

We conclude that the Green's kernel of a noncompact harmonic manifold (X, g) has the form

$$G(x, y) = \tilde{G}(d(x, y)) = \frac{1}{\omega_n} \int_{d(x, y)}^\infty \frac{dt}{f(t)}.$$

Let $p_0 \in X$ be a fixed reference point. We define

$$\Sigma_{p_0} := \{\sigma = (x_m) \subset X \mid d(x_m, p_0) \rightarrow \infty, \frac{G(x, x_m)}{G(p_0, x_m)} \rightarrow K_\sigma(x)\},$$

where the convergence $\frac{G(x, x_m)}{G(p_0, x_m)} \rightarrow K_\sigma(x)$ is meant uniformly on all compact subsets of X . Let $\sigma = (x_m), \sigma' = (x'_m) \in \Sigma$. We call $\sigma \sim \sigma'$ if and only if $K_\sigma = K_{\sigma'}$.

Definition 13.1. *The Martin boundary $\partial_\Delta^{p_0} X$ is defined as*

$$\partial_\Delta^{p_0} X = \Sigma_{p_0} / \sim$$

$\partial_\Delta^{p_0} X$ carries a metric, defined by

$$d_\Delta^{p_0}(\sigma, \sigma') := - \sup_{x \in B(p_0, 1)} |K_\sigma(x) - K_{\sigma'}(x)|.$$

EXAMPLE (Martin Boundary of Euclidean space) Assume that $X = \mathbb{R}^n$ with $n \geq 3$, i.e., X is the flat Euclidean space. Then the Green's kernel is given by

$$G(x, y) = \frac{1}{(n-2) \omega_n |x-y|^{n-2}}.$$

Let $p_0 = 0$. We have

$$\frac{G(x, x_m)}{G(p_0, x_m)} = \left(\frac{|x_m|}{|x - x_m|} \right)^{n-2}.$$

Now, $d(x_m, p_0) \rightarrow \infty$ means that $|x_m| \rightarrow \infty$. Since

$$|x_m| - |x| \leq |x - x_m| \leq |x_m| + |x|,$$

we conclude that, for $|x_m| \rightarrow \infty$,

$$\lim \frac{G(x, x_m)}{G(p_0, x_m)} = 1,$$

i.e., $\partial_\Delta^0 \mathbb{R}^n$ consists of a single point, the constant harmonic function 1.

Since $\partial_B^0 \mathbb{R}^n$ is a topological sphere of dimension $n-1$, we have $\partial_B^0 \mathbb{R}^n \neq \partial_\Delta^0 \mathbb{R}^n$.

However, Busemann boundary and Martin boundary agree for non-flat noncompact harmonic spaces.

Theorem 13.2. *Let (X, g) be a noncompact harmonic manifold with $h > 0$. Then we have*

$$\partial_B^{p_0} X = \partial_\Delta^{p_0} X$$

as topological spaces.

Proof. We first show $\partial_{p_0} X = \partial_\Delta X$ as sets.

(a) Our first goal is: If $\xi \in \partial_B^{p_0} X$, and (x_n) is a sequence in X such that $x_n \rightarrow \xi$ in the Busemann topology, then we have the following uniform convergence on compacta:

$$(13.1) \quad \frac{G(x, x_n)}{G(p_0, x_n)} \rightarrow e^{-h\xi(x)}.$$

This implies that (x_n) is also convergent to a point in the Martin boundary, and we obtain a canonical injective map

$$\partial_B^{p_0}(X) \rightarrow \partial_\Delta^{p_0}(X),$$

$\xi \mapsto (x_n)$, where (x_n) is any sequence with $x_n \rightarrow \xi$ in the Busemann topology.

Proof of (13.1): We first prove: Let I be a compact interval. Then

$$\frac{\int_{s+a}^{\infty} \frac{dt}{f(t)}}{\int_s^{\infty} \frac{dt}{f(t)}} \rightarrow e^{-ha}$$

uniformly on I as $s \rightarrow \infty$. To prove this consider

$$\psi_a(u) = \int_{\frac{1}{u}+a}^{\infty} \frac{dt}{f(t)}.$$

Then $\lim_{u \rightarrow 0} \psi_a(u) = 0$ and by the mean value theorem we obtain

$$(*) \quad \frac{\psi_a(u)}{\psi_0(u)} = \frac{\psi_a(0) - \psi_a(u)}{\psi_0(0) - \psi_0(u)} = \frac{\psi'_a(x)}{\psi'_0(x)}$$

for some $x \in (0, u)$. Since

$$\psi'_a(x) = \frac{1}{f(\frac{1}{x} + a)x^2}$$

we obtain

$$\left| \frac{\psi_a(u)}{\psi_0(u)} - e^{-ha} \right| = \left| \frac{f(\frac{1}{x})}{f(\frac{1}{x} + a)} - e^{-ha} \right|$$

for some $x \in (0, u)$. Using 12.1 we obtain that $\frac{\psi_a(u)}{\psi_0(u)}$ converges on compact intervals uniformly to e^{-ha} as $u \rightarrow 0$ which is equivalent to the assertion above.

Let $h_y(x) = \frac{G(x,y)}{G(p_0,y)}$. Since $G(x,y) = \frac{1}{\omega_n} \int_{d(x,y)}^{\infty} \frac{dt}{f(t)}$, we have

$$h_y(x) = \frac{\int_{d(x,y)}^{\infty} \frac{dt}{f(t)}}{\int_{d(p_0,y)}^{\infty} \frac{dt}{f(t)}}.$$

Let $x_n \rightarrow \xi \in \partial_B^{p_0} X$ in the Busemann topology. By definition this implies that, for each compact subset $K \subset X$ and all $\epsilon > 0$, there exists $n_0(\epsilon, K) > 0$, such that for all $n \geq n_0(\epsilon, K)$ we have

$$|d(x, x_n) - d(p_0, x_n) - \xi(x)| \leq \epsilon$$

for all $n \geq n_0(\epsilon, K)$ and $x \in K$. In particular, we have for all $n \geq n_0(\epsilon, K)$ and $x \in K$

$$d(p_0, x_n) + \xi(x) - \epsilon \leq d(x, x_n) \leq d(p_0, x_n) + \xi + \epsilon.$$

Since $\frac{1}{f(t)} > 0$, we conclude that

$$\frac{\int_{d(p_0, x_n) + \xi(x) - \epsilon}^{\infty} \frac{dt}{f(t)}}{\int_{d(p_0, x_n)}^{\infty} \frac{dt}{f(t)}} \leq h_{x_n}(x) \leq \frac{\int_{d(p_0, x_n) + \xi(x) + \epsilon}^{\infty} \frac{dt}{f(t)}}{\int_{d(p_0, x_n)}^{\infty} \frac{dt}{f(t)}}$$

Since ξ is bounded on K we can choose for each $\epsilon > 0$ (using $(*)$) a number $n_1(\epsilon, K)$, such that for all $n \geq n_1(\epsilon, K)$,

$$e^{-h(\xi(x) + \epsilon)} \leq h_{x_n}(x) \leq e^{-h(\xi(x) - \epsilon)}.$$

This implies, that

$$\lim_{n \rightarrow \infty} h_{x_n}(x) = e^{-h\xi(x)},$$

uniformly on each compact set. Therefore,

$$\lim_{n \rightarrow \infty} \frac{G(x_n, x)}{G(x_n, p_0)} = e^{-h\xi(x)}$$

converges uniformly on compact sets and (x_n) converges to a point in the Martin boundary.

(b) Let $\sigma = (x_n)$ be convergent to a point in the Martin boundary, i.e. $d(x_n, p_0) \rightarrow \infty$ and $\frac{G(x_n, x)}{G(x_n, p_0)} \rightarrow K_\sigma$, uniformly on compacta. Since $x_n \in X \subset \overline{B^{p_0}(X)}$ and $\overline{B^{p_0}(X)}$ is compact, there exists a subsequence $x_{n_j} \in X$ such that $x_{n_j} \rightarrow \xi \in \overline{B^{p_0}(X)}$, in the Busemann topology. Since $d(x_n, p_0) \rightarrow \infty$, $\xi \in \partial_B^{p_0} X$ we obtain that

$$\frac{G(x_{n_j}, x)}{G(x_{n_j}, p_0)} \rightarrow e^{-h\xi(x)}.$$

Therefore, $K_\sigma = e^{-h\xi}$. Let (x'_{n_j}) be an arbitrary subsequence of (x_n) which is convergent in the Busemann topology to a point $\xi' \in \partial_B^{p_0} X$. Then, from $\frac{G(x_n, x)}{G(x_n, p_0)} \rightarrow K_\sigma$, we conclude that

$$e^{-h\xi} = e^{-h\xi'},$$

i.e., $\xi_0 = \xi'$. Hence (x_n) is convergent in the Busemann topology. This shows that the above map $\partial_B^{p_0}(X) \rightarrow \partial_\Delta^{p_0}(X)$ is also surjective.

(c) Assume $\xi_n \in \partial_B^{p_0}(X)$ converges to $\xi \in \partial_B^{p_0}(X)$, i.e., we have uniform convergence of $\xi_n \rightarrow \xi$, on each compact subset $K \subset X$. In particular,

$$\sup_{x \in B_1(x_0)} |\xi_n(x) - \xi(x)| \rightarrow 0,$$

and, consequently, $\sup_{x \in B_1(x_0)} |e^{-h\xi_n(x)} - e^{-h\xi(x)}| \rightarrow 0$. This means that a convergent sequence ξ_n in the Busemann topology is also convergent in the Martin boundary topology.

Therefore, we obtain that the map $\partial_B^{p_0}(X) \rightarrow \partial_\Delta^{p_0}(X)$ with $\xi \rightarrow (x_n)$ where (x_n) is a sequence converging to ξ , is a continuous bijection. Since $\partial_B^{p_0}(X)$ is compact, this map is a homeomorphism. \square

14. REPRESENTATION OF BOUNDED HARMONIC FUNCTIONS

Let (X, g) be a simply connected noncompact harmonic manifold. One of the merits of the Martin boundary theory, which we like to recall, is the representation of bounded harmonic functions (see e.g. [Wo] for more details).

For $p_0 \in X$ define the set

$$H_{p_0}(X) = \{F \in C^2(X) \mid \Delta F = 0, F(p_0) = 1, F \geq 0\}$$

of normalized positive harmonic functions on X . This set is convex and compact with respect to the topology of uniform convergence on compact subsets. We call an element $F \in H_{p_0}(X)$ *minimal*, if for any two function $F_1, F_2 \in H_{p_0}(X)$ with

$$F = \lambda F_1 + (1 - \lambda) F_2$$

we have $F_1 = F_2$. Note, that the minimal set is contained in the Martin boundary. Therefore,

$$\partial_{\Delta, \min}^{p_0}(X) := \{K_\sigma \mid \sigma \in \partial_\Delta^{p_0}(X), K_\sigma \text{ is minimal}\}$$

is the set of minimal elements in $H_{p_0}(X)$. Using Choquet theory we obtain for each $F \in H_{p_0}(X)$ a unique probability measure ν_F on $\partial_{\Delta, \min}^{p_0}(X)$ with

$$F(x) = \int_{\partial_{\Delta, \min}^{p_0}(X)} K_\sigma(x) d\nu_F(\sigma).$$

If we do not insist in probability measures, we can represent in this way positive harmonic functions which are not normalized as well. If F, G are positive harmonic functions with $F \leq G$ then $G - F$ is a positive harmonic function and

$$\mu_G = \nu_F + \nu_{G-F} \leq \nu_F.$$

This implies that ν_F is absolutely continuous to ν_G . Of fundamental importance is the *harmonic measure* $\nu_{p_0} := \nu_1$, i.e the unique probability on $\partial_{\Delta, \min}^{p_0}(X)$ defined by

$$1 = \int_{\partial_{\Delta, \min}^{p_0}(X)} K_\sigma(x) d\nu_{p_0}(\sigma).$$

Using this measure, one can represent all bounded harmonic functions on X in the following way. Consider first a harmonic function F with $0 \leq F \leq M$ for some $M \geq 0$. This implies $\nu_F \leq M\nu_{p_0}$. In particular the Radon-Nikodym derivative

$$\varphi_F := \frac{d\nu_F}{d\nu_{p_0}}$$

exists and defines an element in $L^1(\partial_{\Delta, \min}^{p_0}(X))$ with $0 \leq \varphi_F \leq M$. Hence,

$$F(x) = \int_{\partial_{\Delta, \min}^{p_0}(X)} K_\sigma(x) d\nu_F(\sigma) = \int_{\partial_{\Delta, \min}^{p_0}(X)} K_\sigma(x) \varphi_F(\sigma) d\nu_{p_0}(\sigma).$$

Now, consider an arbitrary bounded harmonic function F and assume that $-M \leq F \leq M$ for some positive number M . Then

$$F(x) + M = \int_{\partial_{\Delta, \min}^{p_0}(X)} K_\sigma(x) \varphi_{F+M}(\sigma) d\nu_{p_0}(\sigma),$$

where $0 \leq \varphi_{F+M} \leq 2M$. Hence,

$$\begin{aligned}
F(x) &= \int_{\partial_{\Delta, \min}^{p_0}(X)} K_\sigma(x) \varphi_{F+M}(\sigma) d\nu_{p_0}(\sigma) - \int_{\partial_{\Delta, \min}^{p_0}(X)} K_\sigma(x) M d\nu_{p_0}(\sigma) \\
&= \int_{\partial_{\Delta, \min}^{p_0}(X)} K_\sigma(x) (\varphi_{F+M}(\sigma) - M) d\nu_{p_0}(\sigma) \\
&= \int_{\partial_{\Delta, \min}^{p_0}(X)} K_\sigma(x) \varphi_F(\sigma) d\nu_{p_0}(\sigma),
\end{aligned}$$

where $-M \leq \varphi_F := \varphi_{F+M} - M \leq M$. In particular, the following theorem holds (see also [Wo, Theorem (24.12)]):

Theorem 14.1. *Denote by $L^\infty(\partial_{\Delta, \min}^{p_0}(X))$ the set of bounded L^1 -functions on $\partial_{\Delta, \min}^{p_0}(X)$, and by $H^\infty(X)$ the set of bounded harmonic functions on X . Then the map $H : L^\infty(\partial_{\Delta, \min}^{p_0}(X)) \rightarrow H^\infty(X)$ with*

$$H(\varphi)(x) := \int_{\partial_{\Delta, \min}^{p_0}(X)} K_\sigma(x) \varphi(\sigma) d\nu_{p_0}(\sigma)$$

defines a linear isomorphism.

Now let (X, g) be a harmonic manifold with $h > 0$. Then as we have shown above (see Theorem 13.2), the Martin boundary $\partial_\Delta^{p_0}(X)$ and the Busemann boundary $\partial_B^{p_0}(X)$ are isomorphic and, using the identification $\sigma \rightarrow \xi$, we have $K_\sigma = e^{-h\xi}$. As has been observed by Zimmer [Zi3], all functions $e^{-h\xi}$ are minimal in $H_{p_0}(X)$. Let μ_{p_0} be the visibility measure with respect to p_0 introduced above. We have that

$$1 = \int_{\partial_B^{p_0}(X)} e^{-h\xi} d\mu_{p_0}$$

which by the discussion above implies that μ_{p_0} is the harmonic measure on $\partial_B^{p_0}(X)$.

Related to the Martin representation is the *Dirichlet problem at infinity*, which deals with the following question. Let $\varphi : \partial_B^{p_0}(X) \rightarrow \mathbb{R}$ be a continuous function. Is there a harmonic function F on X such that

$$\lim_{x \rightarrow \xi} F(x) = \varphi(\xi)?$$

By the maximal principle, F is unique if it exists. A natural candidate for the solution is the function H_φ defined by the integral presentation:

$$H_\varphi(x) = \int_{\partial_B^{p_0}(X)} \varphi(\xi) d\mu_x(\xi) = \int_{\partial_B^{p_0}(X)} \varphi(\xi) e^{-h\xi(x)} d\mu_{p_0}(\xi).$$

Obviously, H_φ is the solution of the Dirichlet problem if and only if

$$\lim_{x \rightarrow \xi} \mu_x = \delta_\xi,$$

where δ_ξ is the Dirac measure at $\xi \in \partial_\Delta^{p_0}(X)$ and the limit is taken in the weak topology of Borel probability measure on $\partial_\Delta^{p_0}(X)$. We will see in Chapter 18 that this holds if (X, g) has purely exponential volume growth.

Part 2. Noncompact harmonic manifolds with purely exponential volume growth

From now on, all harmonic manifolds (X, g) under consideration are assumed to be noncompact, connected and simply connected, and of purely exponential volume growth. As shown in the paper [Kn3] by the first author, purely exponential volume growth, geometric rank 1, Gromov hyperbolicity and the Anosov-property of the geodesic flow (with respect to the Sasaki-metric) are equivalent for harmonic spaces (X, g) . Furthermore, nonpositive curvature or more generally no focal points implies any of the above conditions.

This part covers the following new results for harmonic manifolds with purely exponential volume growth:

- (1) Agreement of the geometric boundary and the Busemann boundary as topological spaces.
- (2) Solution of the Dirichlet Problem at infinity, and an explicit integral presentation using the visibility measures at infinity. The solution of the Dirichlet Problem follows also from the general theory of [Anc1, Anc2] for Gromov hyperbolic spaces (which is related to earlier work for spaces of negative curvature by [AnSch], see also [SchY, Chapter II]). However, in the case of harmonic spaces, these results can be deduced in much more direct and geometric way.
- (3) Polynomial volume growth of all horospheres. As an application, we also prove a mean value property of bounded harmonic functions at infinity. This latter result follows from a modification of the arguments given in [CaSam] for negatively curved asymptotically harmonic spaces.

15. GROMOV HYPERBOLICITY

We start this chapter by introducing the Gromov product.

Definition 15.1. *Let (X, d) be a metric space and $x_0 \in X$ a reference point. The Gromov product $(x, y)_{x_0}$ of $x, y \in X$ is defined as*

$$(x, y)_{x_0} = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y))$$

Note that the Gromov product $(x, y)_{x_0}$ is non-negative, by the triangle inequality. A metric space (X, d) is called a *geodesic space*, if any two points $x, y \in X$ can be connected by a geodesic, i.e., if there exists a curve $\sigma_{xy} : [0, d(x, y)] \rightarrow X$ connecting x and y , such that $d(\sigma_{xy}(s), \sigma_{xy}(t)) = |t - s|$ for all $s, t \in [0, d(x, y)]$.

Lemma 15.2. *Let (X, d) be a geodesic space. Then we have, for $x, y, x_0 \in X$,*

$$(x, y)_{x_0} \leq d(x_0, \sigma_{xy}).$$

Proof. Consider $x' \in \sigma_{xy}$ such that $d(x_0, \sigma_{xy}) = d(x_0, x')$. Then

$$\begin{aligned} (x, y)_{x_0} &= \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y)) \\ &= \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, x') - d(x', y)) \\ &\leq \frac{1}{2}((d(x', x_0) + d(x', x_0)) = d(x_0, \sigma_{xy})) \end{aligned}$$

□

Definition 15.3. A geodesic space (X, d) is called δ -hyperbolic if every geodesic triangle Δ is δ -thin, i.e., every side of Δ is contained in the union of the δ -neighborhoods of the other two sides. If a geodesic space (X, d) is δ -hyperbolic for some $\delta \geq 0$, we call (X, d) a Gromov hyperbolic space.

Let us recall the following two general results for Gromov hyperbolic spaces. Note that one of the inequalities in Proposition 15.5 was stated in Lemma 15.2.

Proposition 15.4. (see [CDP, Prop. 1.3.6]) Let (X, d) be a δ -hyperbolic space. Then we have for all $x_0, x, y, z \in X$:

$$(x, y)_{x_0} \geq \min\{(x, z)_{x_0}, (y, z)_{x_0}\} - 8\delta.$$

Proposition 15.5. (see [CDP, Prop. 3.2.7]) Let (X, d) be a δ -hyperbolic space. Then we have

$$(x, y)_{x_0} \leq d(x_0, \sigma_{xy}) \leq (x, y)_{x_0} + 32\delta.$$

Now assume that X is a harmonic manifold. We have seen in proposition 11.4 that the Busemann boundary $\partial_B^{p_0} X$ of X with respect to $p_0 \in X$ can be identified with $S_p X$. The map $\varphi_p^{p_0} : S_p X \rightarrow \partial_B^{p_0} X$ with $\varphi_p^{p_0}(v) := b_v - b_v(p_0)$ is a homeomorphism. A sequence $x_n = \exp_p(t_n v_n) \in X$ with $v_n \in S_p X$ and $t_n \geq 0$ converges to a point $\xi \in \partial_B^{p_0} X$ if and only if $t_n \rightarrow \infty$ and there exists $v \in S_p X$ with $v_n \rightarrow v$. In particular, ξ is given by $b_v - b_v(p_0)$. Hence, a sequence x_n converges in the Busemann topology to infinity if and only if $d(x_n, p) \rightarrow \infty$ and $\angle_p(x_n, x_m) \rightarrow 0$ for $n, m \rightarrow \infty$.

Assuming additionally that X has purely exponential volume growth and therefore is Gromov hyperbolic we show that x_n converges to infinity if and only if

$$\lim_{n, m \rightarrow \infty} (x_n, x_m)_p = \infty.$$

We note that this is used for general Gromov hyperbolic manifolds as a definition for convergence to infinity. (see [BS, Section 2.2]).

The following result is the main result of this chapter.

Theorem 15.6. Let X be harmonic manifold with purely exponential volume growth, $p \in X$ and $\{x_n\}$ be a sequence in X . The following are equivalent.

- (a) The sequence $\{x_n\}$ converges in the Busemann topology to infinity, i.e. $d(x_n, p) \rightarrow \infty$ and $\angle_p(x_n, x_m) \rightarrow 0$ for $n, m \rightarrow \infty$.
- (b) $(x_n, x_m)_p \rightarrow \infty$ for $n, m \rightarrow \infty$.

Proof. (b) \Rightarrow (a): It was shown in [Kn3] that X is δ -hyperbolic for some $\delta \geq 0$. Let $(x_n, x_m)_p \rightarrow \infty$. We know from Lemma 15.2 that $d(p, x_n), d(p, x_m) \geq (x_n, x_m)_p$, which shows that $d(p, x_n) \rightarrow \infty$ as $n \rightarrow \infty$. It remains to show that $\angle_p(x_n, x_m) \rightarrow 0$. Let U_{px_n}, U_{px_m} be δ -tubes around the geodesic arcs σ_{px_n} and σ_{px_m} . Then the geodesic $\sigma_{x_n x_m}$ must contain a point $p_1 \in U_{px_n} \cap U_{px_m}$. We conclude from Lemma 15.2 that

$$d(p_1, p) \geq d(\sigma_{x_n x_m}, p) \geq (x_n, x_m)_p.$$

Let γ_1 and γ_2 be the shortest curves connecting p_1 with σ_{px_n} and σ_{px_m} at the points y_n and y_m , see Figure 5. Then $d(p_1, y_n), d(p_1, y_m) \leq \delta$, which implies $d(y_n, y_m) \leq 2\delta$ and

$$d(y_n, p), d(y_m, p) \geq (x_n, x_m)_p - \delta.$$

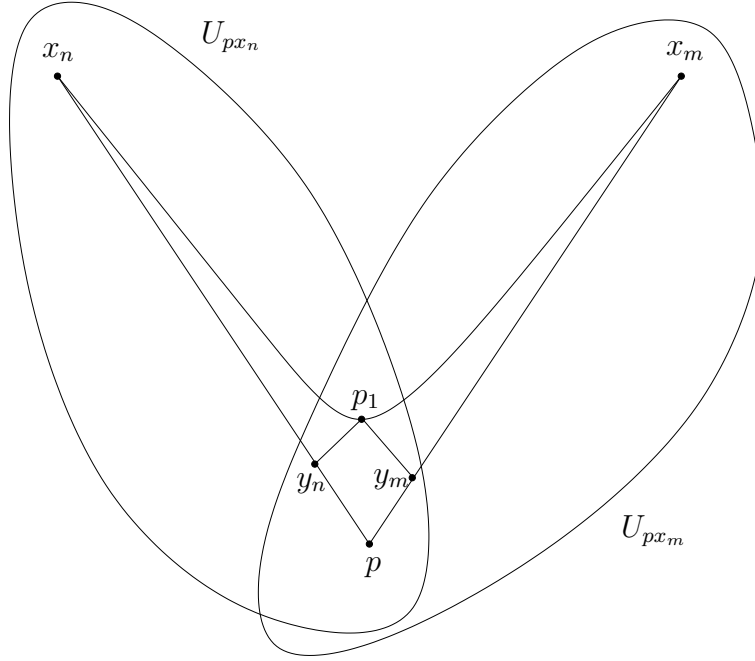


FIGURE 5. Illustration of the proof of (b) \Rightarrow (a) in Theorem 15.6

We assume, without loss of generality, that $d(y_n, p) \geq d(y_m, p)$. Let $z_m \in \sigma_{px_m}$ be such that $d(p, z_m) = d(p, y_n)$. This implies that

$$d(y_n, p) = d(z_m, p) \geq (x_n, x_m)_p - \delta.$$

Since

$$d(y_m, p) \leq d(z_m, p) = d(y_n, p) \leq d(y_m, p) + d(y_n, y_m) \leq d(y_m, p) + 2\delta,$$

and since y_m, z_m lie on the same geodesic arc σ_{px_m} , we have $d(y_m, z_m) \leq 2\delta$. This implies that

$$d(z_m, y_n) \leq d(y_m, y_n) + d(z_m, y_n) \leq 2\delta + 2\delta = 4\delta.$$

Using Corollary 3.4, we conclude that

$$4\delta \geq \text{length}(\sigma_{y_n z_m}) \geq a(d(y_n, p)) \angle_p(x_n, x_m).$$

Since $d(y_n, p) \rightarrow \infty$, we also have $a(d(y_n, p)) \rightarrow \infty$, which implies that $\angle_p(x_n, x_m) \rightarrow 0$.

(a) \Rightarrow (b): Assume $\angle_p(x_n, x_m) \rightarrow 0$ and $d(x_n, p) \rightarrow \infty$ for $n, m \rightarrow \infty$. For all $R > 0$, there exists $n_0(R) \geq 0$, such that for all $n, m \geq n_0(R)$:

$$(15.1) \quad d(p, x_n), d(p, x_m) \geq R \quad \text{and} \quad d(c_{px_n}(R), c_{px_m}(R)) \leq 1,$$

since $\angle_p(x_n, x_m) \rightarrow 0$ for $n, m \rightarrow \infty$. Note that the constant $n_0(R)$ does not depend on p , but only on the values $d(p, x_n)$ and $\angle_p(x_n, x_m)$, since X has a uniform lower curvature bound.

We show now the following: *The geodesic arc $\sigma_{x_n x_m}$ has empty intersection with the open ball $B_{R-\frac{1}{2}}(p)$ for all $n, m \geq n_0(R)$.*

If $\sigma_{x_n x_m} \cap B_R(p) = \emptyset$, there is nothing to prove. If $\sigma_{x_n x_m} \cap B_R(p) \neq \emptyset$, there exists a first $t_0 > 0$ and a last $t_1 > 0$ such that

$$q_1 = \sigma_{x_n x_m}(t_0), q_2 = \sigma_{x_n x_m}(t_1) \in S_R(p),$$

where $S_R(p)$ denotes the sphere of radius $R > 0$ around p (see Figure 6). Then we have

$$d(q_1, q_2) = l(\sigma_{x_n x_m}) - d(x_n, q_1) - d(x_m, q_2).$$

Using (15.1), we have

$$\begin{aligned} l(\sigma_{x_n x_m}) &\leq d(x_n, \sigma_{px_n}(R)) + d(\sigma_{px_n}(R), \sigma_{px_m}(R)) + d(\sigma_{px_m}(R), x_m) \\ &\leq d(x_n, \sigma_{px_n}(R)) + d(x_m, \sigma_{px_m}(R)) + 1, \end{aligned}$$

which implies that

$$(15.2) \quad \begin{aligned} d(q_1, q_2) &\leq (d(x_n, \sigma_{px_n}(R)) - d(x_n, q_1)) \\ &\quad + (d(x_m, \sigma_{px_m}(R)) - d(x_m, q_2)) + 1. \end{aligned}$$

Since $d(p, x_n) = R + d(\sigma_{px_n}(R), x_n) \leq d(q_1, x_n) + R$ (by the triangle inequality), we obtain $d(x_n, q_1) - d(x_n, \sigma_{px_n}(R)) \geq 0$, and similarly $d(x_m, q_2) - d(x_m, \sigma_{px_m}(R)) \geq 0$. This, together with (15.2) shows $d(q_1, q_2) \leq 1$. But then the geodesic segment of $\sigma_{x_n x_m}$ between q_1 and q_2 cannot enter the ball $B_{R-\frac{1}{2}}(p)$.

Therefore, we have for all $n, m \geq n_0(R)$,

$$R - \frac{1}{2} \leq d(p, \sigma_{x_n x_m}) \leq (x_n, x_m)_p + 32\delta,$$

using Proposition 15.5. This shows that

$$(x_n, x_m)_p \rightarrow \infty \quad \text{as } n, m \rightarrow \infty.$$

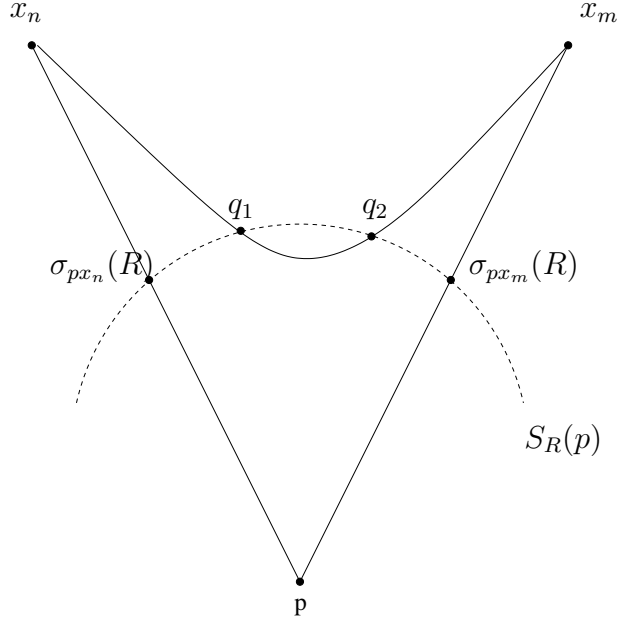


FIGURE 6. Illustration of the proof of $(a) \Rightarrow (b)$ in Theorem 15.6

□

16. THE GEOMETRIC BOUNDARY OF A HARMONIC SPACE WITH PURELY EXPONENTIAL VOLUME GROWTH

This chapter provides a purely self-contained introduction into the geometric boundary $X(\infty)$, based on equivalent geodesic rays, and its associated cone topology for harmonic spaces with purely exponential volume growth.

We like to mention that all results in the later chapters 18, 19 and 20 could have also been formulated in terms of the Busemann boundary $\partial_B X$ instead of the geometric boundary $X(\infty)$, in which case the current chapter as well as the following Chapter 17 would be of no relevance for these later results.

We first recall the classical definition of the geometric boundary.

Definition 16.1. *The geometric boundary $X(\infty)$ of the harmonic space (X, g) consists of all equivalence classes of geodesic rays, where two unit speed geodesic rays $\sigma_1, \sigma_2 : [0, \infty) \rightarrow X$ are equivalent if there exists $C > 0$ such that*

$$d(\sigma_1(t), \sigma_2(t)) \leq C$$

for all $t \geq 0$. The equivalence class of a (unit speed) geodesic ray σ is denoted by $[\sigma]$. The geodesic ray with initial vector $v \in SX$ is denoted by σ_v .

Proposition 16.2. *For every $p \in X$, the map*

$$\begin{aligned}\Phi_p : S_p X &\rightarrow X(\infty), \\ \Phi_p(v) &= [\sigma_v]\end{aligned}$$

is injective.

Proof. This is an immediate consequence of the uniform divergence of geodesics (see Corollary 3.4). \square

REMARK Proposition 16.2 holds in the general context of non-compact harmonic manifolds (without the purely exponential volume growth condition).

Lemma 16.3. *There exists a universal constant $A > 0$, only depending on X , such that for all unit speed geodesic rays $\sigma_1, \sigma_2 : [0, \infty) \rightarrow X$ with $\sigma_1(0) = \sigma_2(0)$ we have*

$$d(\sigma_1(t), \sigma_2(t)) \leq Ad(\sigma_1(T), \sigma_2(T)) \quad \forall 0 \leq t \leq T.$$

Proof. Because of the uniform lower and upper bound on the sectional curvature of X , there exist $A_0 \geq 1$ and $\epsilon > 0$, only depending on X , such that

$$d(\sigma_1(t), \sigma_2(t)) \leq A_0 d(\sigma_1(T), \sigma_2(T)) \quad \forall 0 \leq t \leq T \leq \epsilon.$$

Using again the lower curvature bound on X and Corollary 3.4, we can find for every $R > 0$ a constant $A_1(R) \geq 1$ such that

$$d(\sigma_1(t), \sigma_2(t)) \leq A_1(X, R) d(\sigma_1(T), \sigma_2(T)) \quad \forall \epsilon \leq t \leq T \leq R.$$

We assume now that $R > 0$ is chosen large enough that $R/2$ is greater than a certain universal constant $c > 0$, introduced later and only depending on X . It remains to show that there exists a universal constant $A_2 \geq 1$, only depending on X , such that

$$d(\sigma_1(t), \sigma_2(t)) \leq A_2(X, R) d(\sigma_1(T), \sigma_2(T)) \quad \forall R \leq t \leq T.$$

Let $T > R$, and $c : [0, 1] \rightarrow X$ be a geodesic connecting $\sigma_1(T)$ with $\sigma_2(T)$, written as

$$c(s) = \exp_p r(s)v(s),$$

where $p = \sigma_1(0)$, $r(0) = r(1) = T$ and $v(s) \in S_p X$ for all $0 \leq s \leq 1$. Assume first that there is $s_0 \in (0, 1)$ such that $r(s_0) = d(c(s_0), p) \leq T/2$. Then $d(\sigma_1(T), \sigma_2(T)) \geq T$ and we have

$$d(\sigma_1(t), \sigma_2(t)) \leq 2t \leq 2T \leq 2d(\sigma_1(T), \sigma_2(T)).$$

So we can disregard this case and assume that $r(s_0) = d(c(s_0), p) > T/2$, for all $s_0 \in [0, 1]$. Let $R \leq t = \delta T$, and $c_\delta : [0, 1] \rightarrow X$ be given by $c_\delta(s) = \exp_p \delta r(s)v(s)$. Then

$$\begin{aligned}\left. \frac{d}{ds} \right|_{s=s_0} c_\delta(s) &= D \exp_p (\delta r(s_0)v(s_0)(\delta r'(s_0)v(s_0) + \delta r(s_0)v'(s_0)) \\ &= r'(s_0)c'_{v(s_0)}(\delta r(s_0)) + A_{v(s_0)}(\delta r(s_0))(v'(s_0))\end{aligned}$$

Since $c'_{v(s_0)}(\delta r(s_0)) \perp A_{v(s_0)}(\delta r(s_0))(v'(s_0))$, we obtain

$$\begin{aligned} \left\| \frac{d}{ds} \Big|_{s=s_0} c_\delta(s) \right\|^2 &= (r'(s_0))^2 + \|A_{v(s_0)}(\delta r(s_0))v'(s_0)\|^2 \\ &\leq (r'(s_0))^2 + \|A_{v(s_0)}(\delta r(s_0))A_{v(s_0)}^{-1}(r(s_0))\|^2 \cdot \|A_{v(s_0)}(r(s_0))(v'(s_0))\|^2. \end{aligned}$$

Let $B(t) = A_{v(s_0)}(t)A_{v(s_0)}^{-1}(r(s_0))$. This is an orthogonal Jacobi-Tensor along $c_{v(s_0)}$. Let $w \in (c'_{v(s_0)}(r(s_0)))^\perp$ with $\|w\| = 1$. Then $J(t) = B(t)w$ is a Jacobi field with $J(0) = 0, J(r(s_0)) = w$. Using the Anosov property of the geodesic flow and the theorem in [Bo, p. 107], we conclude the existence of a universal $A_2 \geq 2$, such that

$$(16.1) \quad \|J(t)\| \leq A_2 \|J(r(s_0))\| = A_2$$

for all $t \in [c, r(s_0)]$ with a universal constant $c > 0$, only depending on X . Therefore

$$\|B(t)\| \leq A_2$$

for all $c \leq t \leq r(s_0)$. Note that $\delta r(s_0) > \delta \frac{T}{2} \geq \frac{R}{2}$, and that we assumed earlier that $R/2 > c$. This implies that

$$\begin{aligned} \left\| \frac{d}{ds} \Big|_{s=s_0} c_\delta(s) \right\|^2 &\leq A_2^2 ((r'(s_0))^2 + \|A_{v(s_0)}(r(s_0))v'(s_0)\|^2) \\ &\leq A_2^2 \left\| \frac{d}{ds} \Big|_{s=s_0} c(s) \right\|^2, \end{aligned}$$

which shows that

$$\begin{aligned} \text{length}(c_\delta) &= \int_0^1 \left\| \frac{d}{ds} \Big|_{s=s_0} c_\delta(s) \right\| ds \\ &\leq A_2 \int_0^1 \left\| \frac{d}{ds} \Big|_{s=s_0} c(s) \right\| ds = A_2 d(\sigma_1(T), \sigma_2(T)). \end{aligned}$$

Since c_δ connects $\sigma_1(\delta T)$ with $\sigma_2(\delta T)$, we conclude

$$d(\sigma_1(t), \sigma_2(t)) \leq A_2 d(\sigma_1(T), \sigma_2(T)) \quad \text{for all } R \leq t = \delta T \leq T.$$

□

Proposition 16.4. *For every $p \in X$, the map $\Phi_p : S_p X \rightarrow X(\infty)$ is bijective.*

Proof. In view of Proposition 16.2, it suffices to prove surjectivity of Φ_p . Let $[\sigma] \in X(\infty)$ and $\sigma(0) = q$. Choose $t_n \rightarrow \infty$ and $v_n \in S_p X$ and $s_n \in \mathbb{R}$ such that $\sigma(t_n) = \exp_p(s_n v_n)$. Obviously

$$|t_n - s_n| \leq d(p, q).$$

Then $d(p, \sigma(t_n)) = d(q_n, \sigma(t_n)) = s_n$, where $q_n := \sigma(t_n - s_n)$. By Lemma 16.3, we have

$$\begin{aligned} d(c_{v_n}(t), \sigma(t_n - s_n + t)) &\leq A d(p, q_n) \leq A (d(p, q) + |t_n - s_n|) \\ &\leq (A + 1) d(p, q) \end{aligned}$$

for all $t \in [0, s_n]$. This implies that

$$\begin{aligned} d(c_{v_n}(t), \sigma(t)) &\leq d(c_{v_n}(t), \sigma(t_n - s_n + t)) + d(\sigma(t_n - s_n + t), \sigma(t)) \\ &\leq (A + 1) d(p, q) + |t_n - s_n| \leq (A + 2) d(p, q) \end{aligned}$$

for all $t \in [0, s_n]$. Since $S_p X$ compact, there exists $v_0 \in S_p X$ such that a subsequence of v_n converges to v_0 . We conclude that $d(c_{v_0}(t), \sigma(t)) \leq (A + 2) d(p, q)$ for all $t \geq 0$, i.e., $[\sigma] = [c_{v_0}]$. This shows surjectivity. \square

The next proposition is an easy consequence of the uniform divergence of geodesics: For every distance $d > 0$ and every angle $\epsilon > 0$ there exists a uniform radius \hat{R} such that any two points p, q at distance $\leq d$ and outside any ball of radius \hat{R} will be seen from the center of this ball in an angle $\leq \epsilon$.

Proposition 16.5. *For $d > 0$ and $\epsilon > 0$ there exists $\hat{R} = \hat{R}(d, \epsilon, X) \geq 0$, such that for all $p_0 \in X$ and for all $p, q \notin B_{\hat{R}}(p_0)$ with $d(p, q) \leq d$ we have*

$$\angle_{p_0}(p, q) \leq \epsilon.$$

Proof. Since $a(t) \rightarrow \infty$ as $t \rightarrow \infty$, we can choose $\hat{R} > 0$ such that, for all $t \geq \hat{R}$, we have $\frac{2d}{a(t)} \leq \epsilon$. Let $p, q \notin B_{\hat{R}}(p_0)$ with $d(p, q) \leq d$. Let $v, w \in S_{p_0} X$ and $t_1, t_2 > \hat{R}$ such that $p = c_v(t_1)$ and $q = c_w(t_2)$. We conclude from the triangle inequality that $|t_2 - t_1| \leq d$. Let $q_0 = c_w(t_1) \notin B_{\hat{R}}(p_0)$. Then

$$d(p, q_0) \leq d(p, q) + d(q, q_0) \leq d + |t_2 - t_1| \leq 2d.$$

Then Corollary 3.4 implies that

$$\angle_{p_0}(p, q) = \angle_{p_0}(p, q_0) \leq \frac{d(p, q_0)}{a(t_1)} \leq \frac{2d}{a(t_1)} \leq \epsilon,$$

since $t_1 \geq \hat{R}$. This finishes the proof. \square

The next result states that far out points of a geodesic ray σ , seen from another point p at bounded distance $\leq d$ from $\sigma(0)$, appear under a very small angle.

Proposition 16.6. *Let $d > 0$ be given. Let $\sigma : [0, \infty) \rightarrow X$ be a unit speed geodesic ray with $q := \sigma(0)$. Let $p \in B_d(q) = \{z \in X \mid d(z, q) \leq d\}$. For $T > 0$, let $\sigma_T : [0, d(p, \sigma(T))] \rightarrow X$ be the unit speed geodesic from p to $\sigma(T)$. Then for $\epsilon > 0$ there exists $C = C(X, d, \epsilon) > 0$ such that for all $S, T \geq C$,*

$$\angle_p(\sigma'_S(0), \sigma'_T(0)) \leq \epsilon.$$

Proof. The details of the following proof are illustrated in Figure 7. By [Kn3], X is ρ -Gromov hyperbolic for some $\rho > 0$. Since X has lower curvature bound, we can find $\delta > 0$ such that for all $z \in X$ and for all $v_1, v_2 \in S_z X$ we have

$$(16.2) \quad \angle_z(v_1, v_2) \leq \delta \quad \Rightarrow \quad d(c_{v_1}(\rho + 1), c_{v_2}(\rho + 1)) \leq 1.$$

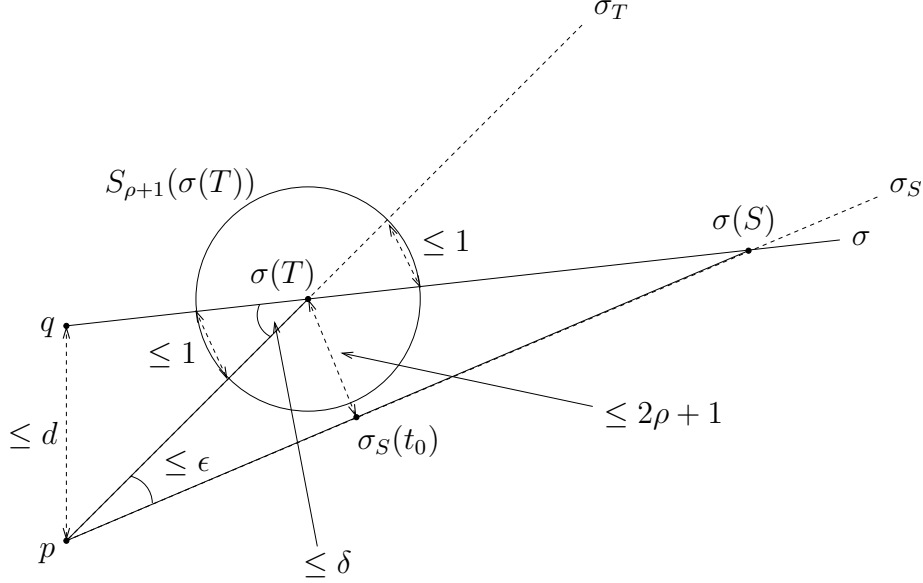


FIGURE 7. Illustration of the proof of Proposition 16.6

Choose

$$C := \max\{\widehat{R}(d, \delta, X), \widehat{R}(2\rho + 1, \epsilon, X) + 2\rho + 1\} + d$$

with $\widehat{R}(d, \epsilon, X)$ as in Proposition 16.5. Without loss of generality, we can assume that $S \geq T$. We know that

$$d(p, \sigma(T)), d(q, \sigma(T)) \geq T - d \geq \widehat{R}(d, \delta, X),$$

and conclude from Proposition 16.5 that

$$\angle_{\sigma(T)}(p, q) \leq \delta.$$

Using (16.2), this implies that

$$d(\sigma_T(d(p, \sigma(T)) \pm (\rho + 1)), \sigma(T \pm (\rho + 1))) \leq 1.$$

Using [Kn3, Cor. 4.5], we conclude that there exists $t_0 > 0$ such that $d(\sigma(T), \sigma_S(t_0)) \leq 2\rho + 1$. (In the case $S \in [T, T + (\rho + 1)]$, we choose $t_0 = d(p, \sigma(S))$ and have $\sigma_S(t_0) = \sigma(S)$.) Since $T \geq C \geq \widehat{R}(2\rho + 1, \epsilon, X) + 2\rho + 1 + d$, we have

$$d(\sigma(T), p) \geq d(\sigma(T), q) - d(p, q) = T - d(p, q) \geq \widehat{R}(2\rho + 1, \epsilon, X) + 2\rho + 1$$

and

$$\begin{aligned} d(\sigma_S(t_0), p) &\geq d(\sigma(T), p) - d(\sigma_S(t_0), \sigma(T)) \\ &\geq \widehat{R}(2\rho + 1, \epsilon, X) + 2\rho + 1 - (2\rho + 1) = \widehat{R}(2\rho + 1, \epsilon, X). \end{aligned}$$

Using Proposition 16.5 again, we conclude that

$$\angle_p(\sigma(T), \sigma_S(t_0)) = \angle_p(\sigma'_T(0), \sigma'_S(0)) \leq \epsilon,$$

finishing the proof of the proposition. \square

This proposition has the following limit version (for $S \rightarrow \infty$).

Corollary 16.7. *Let $d > 0$ be given. Let $\sigma : [0, \infty) \rightarrow X$ be a unit speed geodesic ray with $q := \sigma(0)$. Let $p \in B_d(q) = \{z \in X \mid d(z, q) \leq d\}$ and $v \in S_p X$ such that $[\sigma_v] = [\sigma]$. Then for $\epsilon > 0$ we have*

$$\angle_p(v, \sigma'_T(0)) \leq \epsilon$$

for all $T \geq C(X, d, \epsilon)$ with σ_T and $C(X, d, \epsilon)$ as in Proposition 16.6.

Proof. We only need to show that $\lim_{T \rightarrow \infty} \sigma'_T(0) = v$. Then we obtain for $T \geq C(X, d, \epsilon)$:

$$\angle_p(v, \sigma'_T(0)) \leq \limsup_{S \rightarrow \infty} \angle_p(\sigma'_S(0), \sigma'_T(0)) \leq \epsilon,$$

using Proposition 16.6.

We know from Proposition 16.6 that, for any $T_n \rightarrow \infty$, $\sigma'_{T_n}(0)$ is a Cauchy sequence in $S_p X$. Since $S_p X$ is compact, there exists $v_0 = \lim_{T \rightarrow \infty} \sigma'_T(0)$. So it remains to show that $v_0 = v$. Note that we have

$$T - d(p, q) \leq d(p, \sigma(T)) \leq T + d(p, q).$$

Let $v_1 = -\sigma'(T)$ and $v_2 = -\sigma'_T(d(p, \sigma(T)))$. Figure 8 illustrates the following inequalities.

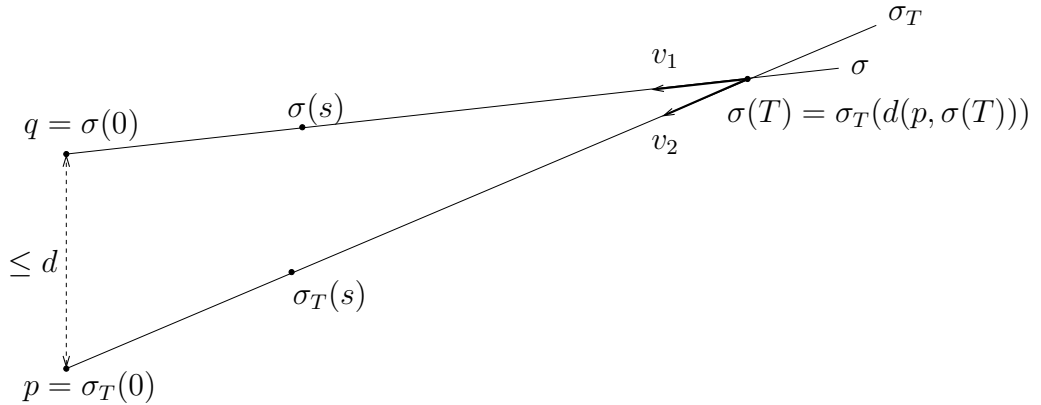


FIGURE 8. Illustration of the proof of Corollary 16.7

Using Lemma 16.3, we have for $0 \leq s \leq T - d(p, q)$:

$$\begin{aligned}
d(\sigma(s), \sigma_T(s)) &= d(\sigma_{v_1}(T - s), \sigma_{v_2}(d(p, \sigma(T)) - s)) \\
&\leq d(\sigma_{v_1}(T - s), \sigma_{v_2}(T - s)) + \\
&\quad d(\sigma_{v_2}(T - s), \sigma_{v_2}(d(p, \sigma(T)) - s)) \\
&\leq A d(\sigma_{v_1}(T), \sigma_{v_2}(T)) + |T - d(p, \sigma(T))| \\
&\leq A (d(q, p) + d(\sigma_{v_2}(d(p, \sigma(T))), \sigma_{v_2}(T))) + d(p, q) \\
&\leq 2A d(p, q) + d(p, q) \leq (2A + 1)d.
\end{aligned}$$

Taking the limit, we conclude that

$$d(\sigma(s), \sigma_{v_0}(s)) \leq (2A + 1)d$$

for all $s \geq 0$, i.e., $[\sigma_{v_0}] = [\sigma]$. Proposition 16.2 implies that $v_0 = v$. \square

For $v_0 \in S_x X$, $R > 0$, $\delta > 0$, we define

$$(16.3) \quad U(v_0, R, \delta) := \{\sigma_v(t) \mid t > R, v \in S_x X \text{ with } \angle_x(v_0, v) < \delta\},$$

which we consider as geometric neighbourhoods of points at the geometric boundary.

Proposition 16.8. *Let $d > 0$ be given. Let $v_0 \in S_p X$ and $v_1 \in S_q X$ with $[\sigma_{v_0}] = [\sigma_{v_1}]$ and $d(p, q) \leq d$. Then for all $R, \delta > 0$ there exist $R' = R'(X, d, R, \delta)$ and $\delta' = \delta'(X, d, R, \delta)$ such that*

$$U(v_1, R', \delta') \subset U(v_0, R, \delta).$$

Proof. We choose

$$R' := \max \left\{ R + d, C(X, d, \frac{\delta}{4}), \widehat{R}(1, \frac{\delta}{4}, X) + d \right\},$$

where \widehat{R} and C are defined as in Propositions 16.5 and 16.6. Let $\sigma = \sigma_{v_1}$. Since $R' \geq C(X, d, \frac{\delta}{4})$, we conclude with Corollary 16.7 that

$$(16.4) \quad \angle_p(v_0, \sigma(R')) \leq \frac{\delta}{4}.$$

Since the curvature of X is bounded from below, there exists $\delta' > 0$, only depending on R' and X , such that

$$\sigma_v(R') \in B_1(\sigma(R')) \quad \forall v \in S_q X \text{ with } \angle_p(v, v_1) < \delta'.$$

Figure 9 illustrates the following arguments.

Let $z \in U(v_1, R', \delta')$. Then $z = \sigma_v(t)$ with $\angle_p(v, v_1) < \delta'$ and $t > R'$.

Since $R' \geq \widehat{R}(1, \frac{\delta}{4}, X) + d$, we have

$$d(p, \sigma(R')), d(p, \sigma_v(R')) \geq \widehat{R}(1, \frac{\delta}{4}, X)$$

and $d(\sigma(R'), \sigma_v(R')) \leq 1$. Using Proposition 16.5, we conclude that

$$(16.5) \quad \angle_p(\sigma(R'), \sigma_v(R')) \leq \frac{\delta}{4}.$$

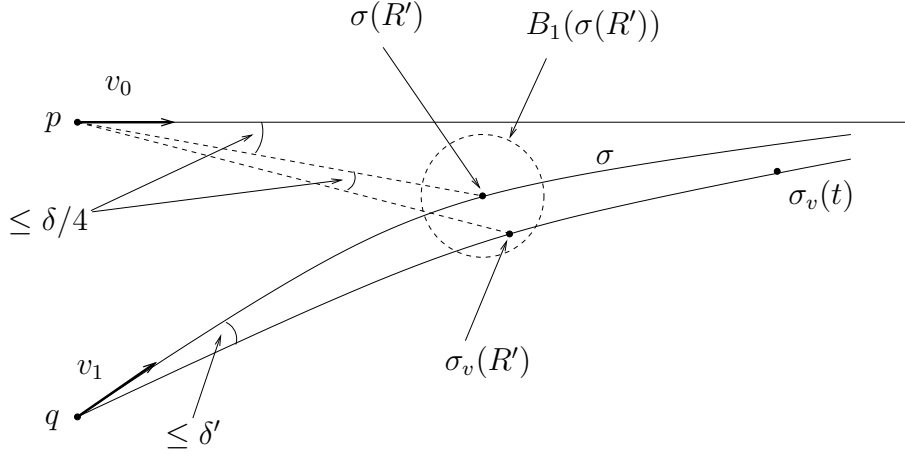


FIGURE 9. Illustration of the proof of Proposition 16.8

Since $t \geq R' \geq C(X, d, \frac{\delta}{4})$, we deduce from Proposition 16.6:

$$(16.6) \quad \angle_p(\sigma_v(R'), \sigma_v(t)) \leq \frac{\delta}{4}.$$

Bringing (16.4), (16.5) and (16.6) together, we obtain

$$\begin{aligned} \angle_p(z, v_0) &\leq \angle_p(\sigma_v(t), \sigma_v(R')) + \angle_p(\sigma_v(R'), \sigma(R')) + \angle_p(\sigma(R'), v_0) \\ &\leq \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{4} < \delta \end{aligned}$$

and

$$d(z, p) \geq d(\sigma_v(t), q) - d(p, q) \geq t - d > R' - d \geq R,$$

i.e., $z \in U(v_0, R, \delta)$. This finishes the proof. \square

Let (X, g) be a noncompact harmonic space *with purely exponential volume growth*. The geometric compactification $\overline{X} = X \cup X(\infty)$ is now the disjoint union of all the points in X and the equivalence classes $[\sigma]$ of unit speed geodesic rays $\sigma : [0, \infty) \rightarrow X$. The above considerations lead to the following natural topology on \overline{X} : A basis of this compact topological space is given by the open balls $U_\epsilon(p) = \{x \in X \mid d(x, p) < \epsilon\}$ with $p \in X$ (neighbourhoods of finite points $p \in X$) and the sets $U(v_0, R, \delta)$, defined in (16.3) (neighbourhoods of the infinite point $[\sigma_{v_0}] \in X(\infty)$). For all $p \in X$, the map

$$\begin{aligned} \Phi_p : \{v \in T_p X \mid \|v\| \leq 1\} = B_p(1) &\longrightarrow X \cup X(\infty), \\ \Phi_p(v) &= \begin{cases} \exp_p \frac{v}{1-\|v\|} & \text{if } \|v\| < 1, \\ [\sigma_v] & \text{if } \|v\| = 1, \end{cases} \end{aligned}$$

is a homeomorphism. The same holds true for the restriction: For all $p \in X$, the map

$$(16.7) \quad \Phi_p : S_p X \rightarrow X(\infty), \quad \Phi_p(v) = [\sigma_v]$$

is a homeomorphism.

17. BUSEMANN FUNCTIONS AND THE GEOMETRIC BOUNDARY

We begin with the following definition:

Definition 17.1. *Let (X, g) be a connected noncompact complete Riemannian manifold. Two unit vectors $v, w \in SX$ are asymptotic directions, if the corresponding geodesic rays $\sigma_v, \sigma_w : [0, \infty) \rightarrow X$ with $\sigma'_v(0) = v$ and $\sigma'_w(0) = w$ stay within bounded distance, i.e., there is a constant $C > 0$ such that*

$$d(\sigma_v(t), \sigma_w(t)) \leq C$$

for all $t \geq 0$. In other words, v and w are asymptotic directions iff σ_v and σ_w define the same equivalence class.

Let (X, g) be a noncompact harmonic space. For all $v \in SX$ and $t \in \mathbb{R}$ let $b_{v,t}(q) = d(q, \sigma_v(t)) - t$. The Busemann function b_v is then defined as

$$b_v(q) = \lim_{t \rightarrow \infty} b_{v,t}(q).$$

Proposition 17.2. *Let (X, g) be a noncompact harmonic space with purely exponential volume growth and $v \in SX$. Then the Busemann function b_v is differentiable, and the vector field $Z(q) = -\text{grad } b_v(q)$ is a vector field of asymptotic directions.*

Proof. We know from Proposition 11.3 that b_v is differentiable and that we have $\text{grad } b_v = \lim_{t \rightarrow \infty} \text{grad } b_{v,t}$, where the convergence is uniform on compact sets.

Let $w = -\text{grad } b_v(q) = \lim_{t \rightarrow \infty} -\text{grad } b_{v,t}(q)$. Since we have $d(\sigma_v(s + (t - d(q, \sigma_v(t))))), \sigma_{-\text{grad } b_{v,t}(q)}(s)) \leq A d(\sigma_v(t - d(q, \sigma_v(t))), q)$, for all $0 \leq s \leq d(q, \sigma_v(t))$, by Lemma 16.3, we conclude

$$\begin{aligned} d(\sigma_v(s), \sigma_{-\text{grad } b_{v,t}(q)}(s)) &\leq |t - d(q, \sigma_v(t))| + A d(\sigma_v(t - d(q, \sigma_v(t))), q) \\ &\leq |t - d(q, \sigma_v(t))| + A(|t - d(q, \sigma_v(t))| + d(p, q)) \\ &\leq d(p, q) + A(d(p, q) + d(p, q)) = (2A + 1)d(p, q), \end{aligned}$$

for all $0 \leq s \leq d(q, \sigma_v(t))$. Keeping $s > 0$ fixed, and taking the limit $t \rightarrow \infty$, we obtain

$$d(\sigma_v(s), \sigma_{-\text{grad } b_v(q)}(s)) \leq (2A + 1)d(p, q) \quad \forall s \geq 0,$$

i.e., v and $Z(q) = -\text{grad } b_v(q)$ are asymptotic directions. \square

Corollary 17.3. *Let (X, g) be a noncompact connected harmonic space with purely exponential volume growth. If $v, w \in SX$ are asymptotic directions, then $b_v - b_w$ is constant.*

Proof. For all $q \in X$, the vectors $-\text{grad } b_v(q), -\text{grad } b_w(q) \in S_q X$ are asymptotic to v and, therefore, asymptotic to each other. Because of Proposition 16.2, we have

$$-\text{grad } b_v(q) = -\text{grad } b_w(q) \quad \forall q \in X.$$

This implies that $\text{grad}(b_v - b_w) \equiv 0$ and, therefore, $b_v - b_w$ must be constant on X . \square

Definition 17.4. Let (X, g) be a noncompact connected harmonic space with purely exponential volume growth. For $p \in X$ and $\xi \in X(\infty)$, we define

$$b_{p,\xi}(q) = b_v(q),$$

where $v \in S_p X$ is given by $[\sigma_v] = \xi$. Note that

$$b_{p,\xi}(p) = 0 \quad \text{and} \quad -\text{grad } b_{p,\xi}(p) = v.$$

The next result states that the Busemann boundary and the geometric boundary agree for noncompact harmonic spaces with purely exponential volume growth.

Theorem 17.5. Let (X, g) be a noncompact connected harmonic space with purely exponential volume growth and $p_0 \in X$ be a reference point. Then there is a canonical homeomorphism $\partial_B^{p_0} X \rightarrow X(\infty)$, given by $b_v \mapsto [\sigma_v]$ for all $v \in S_{p_0} X$.

Proof. We recall from Proposition 11.4(2) that the map $\varphi_{p_0} : S_{p_0} X \rightarrow \partial_B^{p_0} X$, defined by $\varphi_{p_0}(v) = b_v$ is a homeomorphism. We saw at the end of Chapter 16 that the map $\Phi_{p_0} : S_{p_0} X \rightarrow X(\infty)$, $\Phi_{p_0}(v) = [\sigma_v]$ is a homeomorphism (see (16.7)). The canonical homeomorphism introduced in the theorem is the composition of these two homeomorphisms. \square

Let us, finally, return to the visibility measures $\mu_p \in \mathcal{M}_1(\partial_B^{p_0})$, introduced in Chapter 12. Using the canonical homeomorphism $\partial_B^{p_0} X \rightarrow X(\infty)$ in Theorem 17.5, we can view these as probability measures on the geometric boundary $X(\infty)$. Then we have $d\mu_{p_0}([\sigma_v]) = \frac{1}{\omega_n} d\theta_{p_0}(v)$ for all $v \in S_{p_0} X$ and, because of the identity

$$d\mu_p(\xi) = e^{-hb_{p_0,\xi}(p)} d\mu_{p_0}(\xi)$$

for all $\xi \in \partial_B^{p_0}$, we have the identity

$$d\mu_p([\sigma_v]) = e^{-hb_v(p)} d\mu_{p_0}([\sigma_v])$$

for all $v \in S_{p_0} X$ representing $[\sigma_v] \in X(\infty)$. This will become important in Chapter 18 below on the solution of the Dirichlet problem at infinity.

18. SOLUTION OF THE DIRICHLET PROBLEM AT INFINITY

Let (X, g) be a harmonic manifold with purely exponential volume growth. Recall that the Busemann function associated to $v \in S_p X$ is defined as

$$b_v(q) := \lim_{t \rightarrow \infty} d(c_v(t), q) - t.$$

For $v_0 \in S_p X$ and $\delta > 0$, we introduce the cone

$$C(v_0, \delta) = \{c_v(t) \mid t \geq 0, \angle(v_0, v) \leq \delta\}.$$

We already mentioned at the end of Chapter 14 the crucial condition $\lim_{x \rightarrow \xi} \mu_x = \delta_\xi$ to solve the Dirichlet problem at infinity. This abstract condition can be deduced from the geometric fact that any horoball \mathcal{H} , centered at $\xi = c_{v_0}(\infty) \in X(\infty)$, ends up inside any given cone $C(v_0, \delta)$, when being translated to the horoball $\tilde{\mathcal{H}}$ along the stable direction (see the illustration in Figure 10). This is essentially the content of the following proposition. (Note that the horoballs centered at ξ can be described by $\{q \in X \mid b_{v_0}(q) \leq -C\}$, and that these horoballs become smaller and shrink towards the limit point ξ , as $C \in \mathbb{R}$ increases to infinity.)

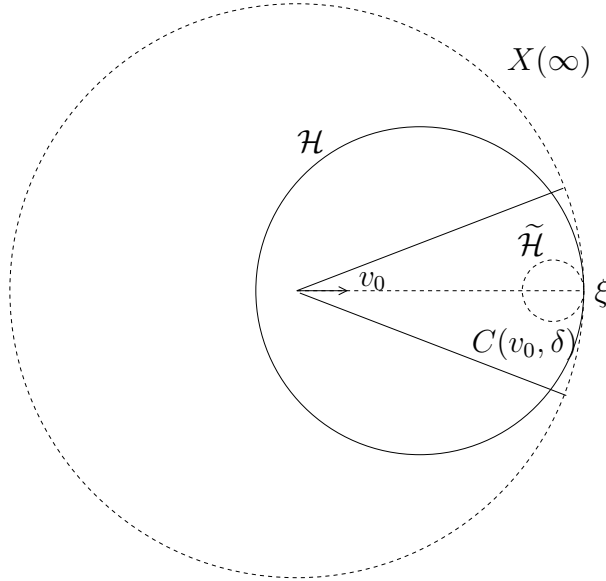


FIGURE 10. Geometric property to guarantee the solution of the Dirichlet problem at infinity

Proposition 18.1. *Let (X, g) be a harmonic space with purely exponential volume growth. Let $v_0 \in S_p X$ and $\delta > 0$. Then there exists a constant $C_1 > 0$, depending only on v_0 and δ , such that*

$$b_{v_0}(q) \geq d(p, q) - C_1 \quad \text{for all } q \in X \setminus C(v_0, \delta).$$

REMARK Proposition 18.1 does not hold if (X, g) is the Euclidean space. In this case, every horoball is a halfspace, which lies never inside a given cone.

Proof. There exists a constant $C_1 > 0$ such that

$$(18.1) \quad 0 \leq 2(c_{v_0}(t), q)_p \leq C_1 \quad \forall t \geq 0 \quad \forall q \in X \setminus C(v_0, \delta).$$

If this were false, then we could find sequences $t_n \geq 0$, $q_n \in X \setminus C(v_0, \delta)$, such that

$$(c_{v_0}(t_n), q_n)_p \rightarrow \infty.$$

This would mean, by Theorem 15.6, that $d(p, q_n) \rightarrow \infty$ and $\angle_p(v_0, q_n) \rightarrow 0$, which is a contradiction to $q_n \in X \setminus C(v_0, \delta)$.

(18.1) means that

$$d(p, q) - (d(c_{v_0}(t), q) - t) \leq C_1 \quad \forall t \geq 0.$$

Taking the limit $t \rightarrow \infty$, we obtain

$$d(p, q) - b_{v_0}(q) = d(p, q) - \lim_{t \rightarrow \infty} (d(c_{v_0}(t), q) - t) \leq C_1,$$

finishing the proof. \square

In fact, we need the following *uniform* modification of Proposition 18.1. Note that in Proposition 18.2 below, $v \in S_p X$ plays the role of v_0 in Proposition 18.1, and every $x \in C(v_0, \frac{\delta}{2})$ satisfies $x \in X \setminus C(v, \frac{\delta}{2})$, because of $\angle_p(v, v_0) \geq \delta$.

Proposition 18.2. *Let (X, g) be a harmonic space with purely exponential volume growth. Let $v_0 \in S_p X$ and $\delta > 0$. Then exists a $C_2 > 0$, depending only on v_0 and δ , such that*

$$b_v(x) \geq d(p, x) - C_2,$$

for all $x \in C(v_0, \frac{\delta}{2})$ and all $v \in S_p X$ with $\angle_p(v, v_0) \geq \delta$.

Proof. There exists a constant $C_2 > 0$ such that

$$0 \leq 2(c_v(t), x)_p \leq C_2 \quad \forall t \geq 0, \quad \forall v \in S_p X \text{ with } \angle_p(v, v_0) \geq \delta, \\ \forall x \in C(v_0, \frac{\delta}{2}),$$

for otherwise, we could find sequences $t_n \geq 0$, $v_n \in S_p X$ with $\angle_p(v_n, v_0) \geq \delta$, and $x_n \in C(v_0, \frac{\delta}{2})$ satisfying

$$(c_{v_n}(t_n), x_n)_p \rightarrow \infty.$$

Using Theorem 15.6, this would imply $d(p, x_n) \rightarrow \infty$ and $\angle_p(v_n, x_n) \rightarrow 0$. But $\angle_p(v_n, x_n) \rightarrow 0$ contradicts to $\angle_p(x_n, v_0) \leq \frac{\delta}{2}$ and $\angle_p(v_n, v_0) \geq \delta$.

Therefore, we have

$$d(x, p) - (d(c_v(t), x) - t) \leq C_2 \quad \forall t \geq 0,$$

which implies, taking $t \rightarrow \infty$, that

$$d(x, p) - b_v(x) \leq C_2 \quad \forall v \in S_p X \text{ with } \angle_p(v, v_0) \geq \delta \text{ and } \forall x \in C(v_0, \frac{\delta}{2}),$$

finishing the proof. \square

Now we state our main result of this chapter, namely, the solution of the Dirichlet problem at infinity in case of purely exponential volume growth.

Theorem 18.3. *Let (X, g) be a harmonic space with purely exponential volume growth. Let $\varphi : X(\infty) \rightarrow \mathbb{R}$ be a continuous function. Then there exists a unique harmonic function $H_\varphi : X \rightarrow \mathbb{R}$ such that*

$$(18.2) \quad \lim_{X \rightarrow \xi} H_\varphi(x) = \varphi(\xi).$$

Moreover, H_φ has the following integral presentation:

$$H_\varphi(x) = \int_{X(\infty)} \varphi(\xi) d\mu_x(\xi),$$

where $\{\mu_x\}_{x \in X} \subset \mathcal{M}_1(X(\infty))$ are the visibility probability measures (originally introduced in Definition 12.1 on $\partial_B X$, and recalled as measures on $X(\infty)$ at the end of Chapter 17).

REMARK Note the differences between the earlier Theorem 14.1 and Theorem 18.3. The earlier theorem states the rather abstract fact that every bounded harmonic function F on X can be represented by a certain integral of a function φ defined on the boundary, using a rather abstract *harmonic measure*. Theorem 14.1 makes no statement about the convergence of $F(x) \rightarrow \varphi(\xi)$, as $x \in X$ converges to ξ . Theorem 18.3 is formulated in the context of continuity, the involved measures are the explicitly given visibility measures μ_x , and it additionally states the crucial convergence $F(x) \rightarrow \varphi(\xi)$.

Proof.

(a) We show first that $\int_{X(\infty)} \varphi(\xi) d\mu_x(\xi)$ is a harmonic function. Let $p \in X$. Then

$$\begin{aligned} \Delta_x \int_{X(\infty)} \varphi(\xi) d\mu_x(\xi) &= \Delta_x \int_{X(\infty)} \varphi(\xi) \frac{d\mu_x}{d\mu_p}(\xi) d\mu_p(\xi) = \\ &= \Delta_x \int_{X(\infty)} \varphi(\xi) e^{-hb_{p,\xi}(x)} d\mu_p(\xi). \end{aligned}$$

Let $K \subset X$ be a compact set. Then $x \mapsto \varphi(\xi) e^{-hb_{p,\xi}(x)}$ is bounded for all $x \in K$ and all $\xi \in X(\infty)$, because of $|b_{p,\xi}(x)| \leq d(p, x)$. Moreover $\Delta_x \varphi(\xi) e^{-hb_{p,\xi}(x)} = 0$ and $b_{p,\xi}(\cdot)$ is smooth, because of $\Delta_x b_{p,\xi} = h$.

Therefore,

$$\Delta_x \int_{X(\infty)} \varphi(\xi) d\mu_x(\xi) = \int_{X(\infty)} \varphi(\xi) \underbrace{\Delta_x e^{-hb_{p,\xi}(x)}}_{=0} d\mu_p(\xi) = 0.$$

(b) Now we prove

$$\lim_{x \rightarrow \xi_0} \int_{X(\infty)} \varphi(\xi) d\mu_x(\xi) = \varphi(\xi_0).$$

Let $\xi_0 = [c_{v_0}]$ with $v_0 \in S_p X$. Without loss of generality, we can assume that $\varphi(\xi_0) = 0$ (by subtracting a constant if necessary). Let $\epsilon > 0$ be given. Then there exists $\delta > 0$, such that

$$|\varphi([c_v])| \leq \epsilon \quad \forall v \in S_p X \text{ with } \angle_p(v_0, v) \leq \delta.$$

We split the integral representing $H_\varphi(x)$ in the following way:

$$\begin{aligned} \omega_n |H_\varphi(x)| \leq & \left| \int_{S_p X \setminus \{v \mid \angle(v_0, v) \leq \delta\}} \varphi([c_v]) e^{-hb_v(x)} d\theta_p(v) \right| + \\ & \left| \int_{\{v \mid \angle(v_0, v) \leq \delta\}} \varphi([c_v]) e^{-hb_v(x)} d\theta_p(v) \right|. \end{aligned}$$

Now, using Proposition 18.2, we obtain

$$\begin{aligned} \omega_n |H_\varphi(x)| & \leq \|\varphi\|_\infty \int_{S_p X \setminus \{v \mid \angle(v_0, v) \leq \delta\}} e^{-h(d(p, x) - C_2)} d\theta_p(v) + \\ & \quad \epsilon \int_{\{v \mid \angle(v_0, v) \leq \delta\}} e^{-hb_v(x)} d\theta_p(v) \leq \\ & \quad \|\varphi\|_\infty \omega_n e^{hC_2} e^{-hd(p, x)} + \underbrace{\epsilon \int_{S_p X} e^{-hb_v(x)} d\theta_p(v)}_{= \int_{S_x X} d\theta_x(v) = \omega_n} \leq \\ & \quad \omega_n (\epsilon + \|\varphi\|_\infty e^{hC_2} e^{-hd(p, x)}). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary and $d(p, x) \rightarrow \infty$ for $x \rightarrow \xi_0$, we conclude that

$$|H_\varphi(x)| \rightarrow 0 \quad \text{for } x \rightarrow \xi_0.$$

(c) Uniqueness of the solution follows from the maximum principle. \square

Let us finish this chapter with an application of Theorem 18.3 (see formulas (18.3) and (18.4) below).

REMARK Obviously, the harmonic function $h_v : X \rightarrow \mathbb{R}$, introduced in Theorem 9.1, has a continuous extension to the compactification $\overline{X} = X \cup X(\infty)$ via

$$h_v(q) = \begin{cases} \mu(r) \langle v, w \rangle & \text{if } q = \exp_p(rw) \in X \text{ with } w \in S_p X, \\ \frac{1}{h} \langle v, w \rangle & \text{if } q = [c_w] \in X(\infty) \text{ with } w \in S_p X. \end{cases}$$

This implies that the harmonic map $F_{E_p} : X \rightarrow B_{\frac{1}{h}}(0)$, introduced in Chapter 10, has an extension as a homeomorphism $\overline{F}_{E_p} : \overline{X} \rightarrow \overline{B_{\frac{1}{h}}(0)}$ with $\overline{F}_{E_p}(p) = 0$.

Since $h_v : \overline{X} \rightarrow \mathbb{R}$ and its restriction on X is harmonic, we know from Theorem 18.3 that

$$h_v(x) = \frac{1}{h} \frac{1}{\omega_n} \int_{S_p X} e^{-hb_w(x)} \langle v, w \rangle d\theta_p(w).$$

On the other hand, we have

$$h_v(x) = \mu(d_p(x)) \cdot \langle v, w_p(x) \rangle,$$

which implies that

$$(18.3) \quad \frac{1}{\omega_n} \int_{S_p X} e^{-hb_w(x)} \langle v, w \rangle d\theta_p(w) = h \cdot \mu(d_p(x)) \cdot \langle v, w_p(x) \rangle,$$

or

$$(18.4) \quad \frac{1}{\omega_n} \int_{S_p X} \underbrace{e^{-hb_w(x)} w}_{\in T_p X} d\theta_p(w) = h \cdot \mu(d_p(x)) \cdot w_p(x).$$

19. HOROSPHERES OF HARMONIC SPACES WITH PURELY EXPONENTIAL VOLUME GROWTH HAVE POLYNOMIAL VOLUME GROWTH

The Anosov property implies for all $v \in SX$ the existence of a splitting

$$T_v SX = E^s(v) \oplus E^u(v) \oplus E^c(v)$$

and constants $a \geq 1$ and $b > 0$ such that for all $\xi \in E^s(v)$

$$(19.1) \quad \|D\phi^t(v)\xi\| \leq a\|\xi\|e^{-bt} \quad \forall t \geq 0.$$

Let $W_v^s \subset SX$ be the corresponding strong stable manifold, i.e., the integral manifold associated to the distribution E^s through $v \in SX$. Its projection $\mathcal{H}_v = \pi W_v^s \subset X$ is a horosphere orthogonal to v . Let $p = \pi(v)$. Consider a curve

$$\xi : [0, 1] \rightarrow W_v^s \quad \xi(0) = v$$

in the strong stable manifold W_v^s such that

$$r \geq \text{length}(\pi \circ \xi) = \int_0^1 \|(\pi \circ \xi)'(s)\| ds.$$

Note that the geodesic flow $\phi^t : SX \rightarrow SX$ induces a bijection $\phi^t : W_v^s \rightarrow W_{\phi^t v}^s$. We apply this to the curve ξ (see Figure 11). Then

$$\pi \circ \phi^t \xi : [0, 1] \rightarrow \mathcal{H}_{\phi_v^t},$$

and, since $(\phi^t \xi)'(s) = D\phi^t(\xi(s))(\xi'(s)) \in E^s(\xi(s))$,

$$\begin{aligned} \text{length}(\pi \circ \phi^t \xi) &= \int_0^1 \|(\pi \circ \phi^t \xi)'(s)\| ds \\ &\leq \int_0^1 \|(\phi^t \xi)'(s)\| ds \stackrel{(19.1)}{\leq} ae^{-bt} \int_0^1 \|\xi'(s)\| ds. \end{aligned}$$

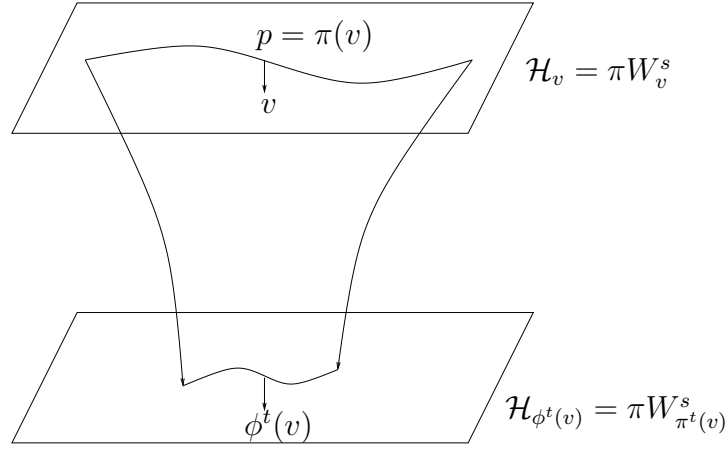


FIGURE 11. Contraction of the geodesic flow on stable horospheres

Since the sectional curvature of a harmonic space is bounded, we conclude that the second fundamental form of horospheres is bounded as well (see the proof of part (A) of Proposition 4.1). Therefore, there exists $C > 0$ such that

$$\|\nabla_{(\pi \circ \xi)'(s)} \xi(s)\| \leq C \|(\pi \circ \xi)'(s)\|.$$

This implies

$$\begin{aligned} \|\xi'(s)\|^2 &= \|(\pi \circ \xi)'(s)\|^2 + \left\| \frac{D}{ds} \xi(s) \right\|^2 = \|(\pi \circ \xi)'(s)\|^2 + \|\nabla_{(\pi \circ \xi)'(s)} \xi(s)\|^2 \\ &\leq (1 + C^2) \|(\pi \circ \xi)'(s)\|^2 \end{aligned}$$

and

$$\text{length}(\pi \circ \phi^t \xi) \leq ae^{-bt} \sqrt{1 + C^2} \text{length}(\pi \circ \xi) \leq ae^{-bt} \sqrt{1 + C^2} r.$$

Hence $\text{length}(\pi \circ \phi^t \xi) \leq 1$ if

$$ae^{-bt} \sqrt{1 + C^2} r \leq 1,$$

or equivalently

$$e^{-bt} \leq \frac{1}{a\sqrt{1 + C^2} r},$$

which means

$$t \geq \frac{\log(a\sqrt{1+C^2} r)}{b} =: t_0.$$

Let $B_{\mathcal{H}}v = \nabla_v \xi$, where ξ is the inward unit normal vector field of the horosphere \mathcal{H} . We know from above that $\|B_{\mathcal{H}}\| \leq C$.

Recall the Gauss equation

$$\langle R(X, Y)Y, X \rangle = \langle R^{\mathcal{H}}(X, Y)Y, X \rangle + \langle X, B_{\mathcal{H}}Y \rangle^2 - \langle B_{\mathcal{H}}X, X \rangle \langle B_{\mathcal{H}}Y, Y \rangle$$

with $X, Y \in T_q \mathcal{H}$. If X, Y are orthonormal, we have

$$|K_{\mathcal{H}}(\text{span}\{X, Y\})| \leq |K(\text{span}\{X, Y\})| + 2\|B_{\mathcal{H}}\|^2 \leq \tilde{C},$$

for some positive constant $\tilde{C} > 0$, since harmonic manifolds have bounded sectional curvature. Therefore the Volume Comparison Theorem yields that any ball of radius 1 in any horosphere has an intrinsic volume bounded by some constant $A > 0$:

$$\text{vol}_{\mathcal{H}}(B_1(q)) \leq A \quad \forall \mathcal{H} \text{ horospheres } \forall q \in \mathcal{H}.$$

This implies that

$$\begin{aligned} \text{vol}_{\mathcal{H}_v}(B_r(p)) &\leq \text{vol}_{\mathcal{H}_v}(\phi^{-t_0}(B_1(\pi \circ \phi^{t_0}(v)))) \\ &\leq e^{ht_0} \text{vol}_{\mathcal{H}_{\phi^{t_0}(v)}}(B_1(\pi \circ \phi^{t_0}(v))) \leq Ae^{ht_0} \\ &= Ae^{\frac{h}{b} \log(a\sqrt{1+C^2} r)} = A(a\sqrt{1+C^2} r)^{h/b}. \end{aligned}$$

This proves the statement in the title of this chapter. \square

20. MEAN VALUE PROPERTY OF HARMONIC FUNCTIONS AT INFINITY

In this chapter, we modify the arguments given in [CaSam] for asymptotic harmonic manifolds of negative curvature. The flow of arguments follows also the arguments given in [KP].

Theorem 20.1. *Let (X, g) be a noncompact harmonic manifold of dimension $n \in \mathbb{N}$ with purely exponential volume growth. Let $\varphi : \overline{X} = X \cup X(\infty) \rightarrow \mathbb{R}$ be continuous, and its restriction $\varphi : X \rightarrow \mathbb{R}$ be harmonic. Let $\xi \in X(\infty)$ and $p_0 \in X$. Let $\mathcal{H} \subset X$ be the horosphere, centered at ξ , containing the point p_0 . Let $K_j \subset \mathcal{H}$ be an exhaustion of \mathcal{H} , such that ∂K_j is smooth and satisfying*

$$\frac{\text{vol}_{n-2}(\partial K_j)}{\text{vol}_{n-1}(K_j)} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Then we have the following "mean value property at infinity":

$$\lim_{j \rightarrow \infty} \frac{\int_{K_j} \varphi(x) dx}{\text{vol}_{n-1}(K_j)} = \varphi(\xi).$$

REMARK Since horospheres have polynomial volume growth, the intrinsic balls of suitably chosen increasing radii satisfy

$$\frac{\text{vol}_{n-2}(\partial B_{\mathcal{H}}(r_j))}{\text{vol}_{n-1}(B_{\mathcal{H}}(r_j))} \rightarrow 0.$$

A suitable choice of sets K_j are regularized spheres, as explained in [KP, p. 665]. But there might be many more increasing sets satisfying this asymptotic isoperimetric property.

Proof. Let $\xi = [c_v]$ with $v \in S_{p_0}X$ and $\mathcal{H} = b_v^{-1}(0)$. Let $\mathcal{H}_t = b_v^{-1}(t)$. Let $\phi_t : X \rightarrow X$ be the flow associated to $\text{grad } b_v = \text{grad } b_{p_0, \xi}$. Then $\phi_t : \mathcal{H}_0 \rightarrow \mathcal{H}_t$. Let $K_j(t) = \phi_t(K_j) \subset \mathcal{H}_t$. Then

$$\text{vol}_{n-1}(K_j(t)) = e^{ht} \text{vol}_{n-1}(K_j).$$

Since X has a lower sectional curvature bound, there exists $C > 0$ such that

$$\text{vol}_{n-2}(\partial K_j(t)) \leq e^{C|t|} \text{vol}_{n-2}(\partial K_j).$$

This implies that, on every compact set $I \subset [0, \infty)$, we have

$$\left\| \frac{\text{vol}_{n-2}(\partial K_j(\cdot))}{\text{vol}_{n-1}(K_j(\cdot))} \right\|_{\infty, I} \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

Define

$$g_j(t) = \frac{\int_{K_j(t)} \varphi(x) dx}{\text{vol}_{n-1}(K_j(t))} \quad \forall t \in \mathbb{R}.$$

Since $\|g_j\|_{\infty} \leq \|\varphi\|_{\infty}$, using diagonal arguments, we find a subsequence g_{j_k} such that $g_{j_k}(t) \rightarrow g(t)$, for all rational t . Since φ is uniformly continuous, we have $g_{j_k} \rightarrow g$ pointwise to a continuous limit.

Next we show that g satisfies

$$(20.1) \quad g'' + hg' = 0,$$

in the distributional sense. Let $\psi \in C_0^{\infty}(\mathbb{R})$ be a test function. Then we have

$$\int_{-\infty}^{\infty} g_j(t)(\psi''(t) - h\psi'(t))dt = \int_{-\infty}^{\infty} \frac{\int_{K_j(t)} \varphi(x) dx}{\text{vol}_{n-1}(K_j(t))} (\psi''(t) - h\psi'(t))dt.$$

Let $\tilde{\varphi} : \mathcal{H} \times (-\infty, \infty) \rightarrow \mathbb{R}$ be defined as $\tilde{\varphi}(x, t) := \varphi(\phi_t(x))$. The tranformation formula yields:

$$\begin{aligned} \int_{K_j(t)} \varphi(x) dx &= \int_{\phi_t(K_j)} \varphi(x) dx = \int_{K_j} \varphi \circ \phi_t(x) \overbrace{\text{Jac } \phi_t(x)}^{e^{ht}} dx = \\ &= e^{ht} \int_{K_j} \tilde{\varphi}(x, t) dx. \end{aligned}$$

Therefore, we have

$$g_j(t) = \frac{1}{\text{vol}(K_j)} \int_{K_j} \varphi(\phi_t x) dx,$$

and

$$\begin{aligned} g_j''(t) + h g_j'(t) &= \frac{1}{\text{vol}(K_j)} \int_{K_j} \frac{d^2}{dt^2} \varphi(\phi_t x) + h \frac{d}{dt} \varphi(\phi_t x) dx \\ &= \frac{1}{\text{vol}(K_j)} \int_{K_j} \Delta_x \varphi(\phi_t x) - \Delta_{\mathcal{H}_t} \varphi(\phi_t x) dx \\ &= \frac{1}{\text{vol}(K_j(t))} \int_{K_j(t)} \underbrace{\Delta_x \varphi(x)}_{=0} - \Delta_{\mathcal{H}_t} \varphi(x) dx \\ &= -\frac{1}{\text{vol}(K_j(t))} \int_{K_j(t)} \Delta_{\mathcal{H}_t} \varphi(x) dx \\ &= \frac{1}{\text{vol}(K_j(t))} \int_{\partial K_j(t)} \langle \text{grad}_{\mathcal{H}_t} \varphi(x), \nu_x \rangle dx, \end{aligned}$$

where ν_x denotes the outward unit vector of $\partial K_j(t) \subset \mathcal{H}_t$. Since $\text{supp} \psi \subset \mathbb{R}$ is compact, we have

$$\begin{aligned} \int_{-\infty}^{\infty} g_j(t) (\psi''(t) - h \psi'(t)) dt &= \int_{-\infty}^{\infty} (g_j''(t) + h g_j'(t)) \psi(t) dt \\ &= \int_{-\infty}^{\infty} \frac{1}{\text{vol}_{n-1}(K_j(t))} \int_{\partial K_j(t)} \langle \text{grad}_{\mathcal{H}_t} \varphi(x), \nu_x \rangle dx \psi(t) dt. \end{aligned}$$

Taking absolute value, we conclude:

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} g_j(t) (\varphi''(t) - h \varphi'(t)) dt \right| \\ &\leq \int_{\text{supp} \psi} \frac{\text{vol}_{n-2}(\partial K_j(t))}{\text{vol}_{n-1}(K_j(t))} \|\text{grad}_X \varphi\|_{\infty} \|\psi\|_{\infty} dt \rightarrow 0, \end{aligned}$$

as $j \rightarrow \infty$, since, by Theorem 8.1:

$$\langle \text{grad}_X \varphi(p), v \rangle = \frac{1}{\text{vol}(B_1(p))} \int_{S_1(p)} \varphi(q) \varphi_v(q) d\mu_1(q) \quad \forall v \in S_p X,$$

which implies

$$\|\operatorname{grad}_X \varphi(p)\| \leq \frac{1}{\operatorname{vol}_n(B_1(p))} \operatorname{vol}_{n-1}(S_1(p)) \|\varphi\|_\infty,$$

i.e.,

$$\|\operatorname{grad}_X \varphi\|_\infty \leq \frac{\operatorname{vol}_{n-1}(S_1(p))}{\operatorname{vol}_n(B_1(p))} \|\varphi\|_\infty.$$

By Lebesgue's dominated convergence, and since $\|g\|_\infty, \|g_j\|_\infty \leq \|\varphi\|_\infty$, we conclude that

$$\int_{-\infty}^{\infty} g(t)(\varphi''(t) - h\varphi'(t))dt = 0,$$

i.e., the continuous function g satisfies (20.1) in the distributional sense. Therefore, g is smooth and satisfies (20.1) in the classical sense, which implies

$$g' + hg = c,$$

for some suitably chosen constant $c \in \mathbb{R}$. The general solution of $g' + hg = c$ is $g(t) = c'e^{-ht} + \frac{c}{h}$ with an arbitrary constant $c' \in \mathbb{R}$. Since g is bounded, we have $g(t) = \frac{c}{h}$.

Let $p_0 \in X$, $v \in S_{p_0}X$ and $\xi = [c_v] \in X(\infty)$. Let $\epsilon > 0$ and $R > 0$. Recall from Proposition 18.1, that we find $t \geq R$ such that

$$(20.2) \quad b_{\xi, p_0}((-\infty, -t]) \subset U(v, R, \epsilon).$$

Continuity of $\varphi : \overline{X} \rightarrow \mathbb{R}$ implies, for every $\epsilon > 0$, that there exists an open neighborhood U of $\xi \in X(\infty)$, such that

$$|\varphi(x) - \varphi(\xi)| < \epsilon \quad \forall x \in U.$$

Choose $t < 0$ negative enough, such that $\mathcal{H}_t = b_{p_0, \xi}^{-1}(t) \subset U$. This is possible because of (20.2). This implies

$$|g_j(t) - \varphi(\xi)| = \frac{\int_{K_j(t)} |\varphi(x) - \varphi(\xi)| dx}{\operatorname{vol}_{n-1}(K_j(t))} \leq \epsilon.$$

Therefore, we have $|g(t) - \varphi(\xi)| \leq \epsilon$ for $t < 0$ negative enough. Since $g(t) = \frac{c}{h}$, i.e., g is constant, we must have $g \equiv \varphi(\xi)$, since $\epsilon > 0$ was arbitrary.

For $t = 0$, we conclude

$$g_j(0) = \frac{\int_{K_j} \varphi(x) dx}{\operatorname{vol}(K_j)} \rightarrow g(0) = \varphi(\xi),$$

as $j \rightarrow \infty$. This finishes the proof. \square

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